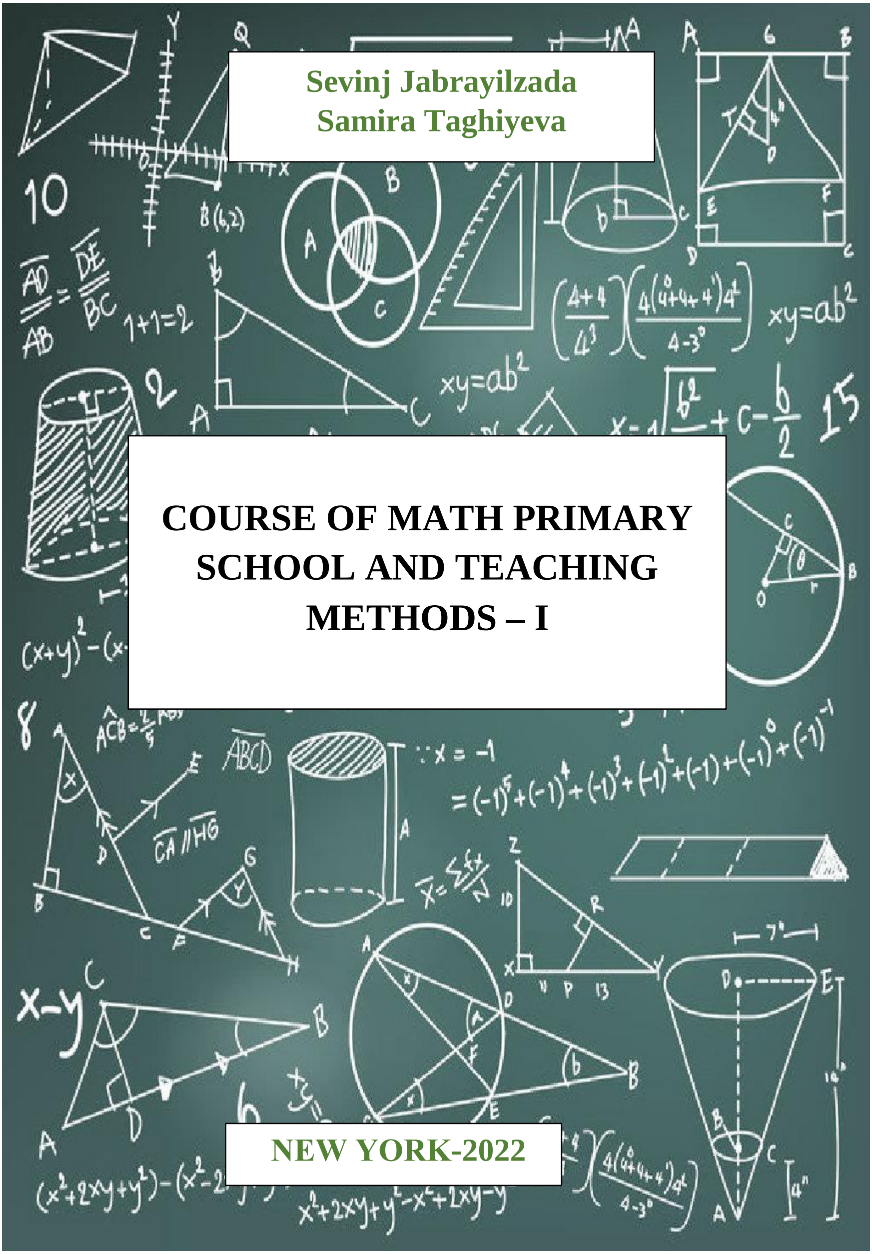

Sevinj Jabrayilzada
Samira Taghiyeva

COURSE OF MATH PRIMARY SCHOOL AND TEACHING METHODS – I

NEW YORK-2022

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This resource is intended for undergraduate students of the Faculty of Elementary Education. The teaching materials can be used by students of mathematics teacher, teachers of higher and secondary schools.

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FOREWORD

These notes in English can be used as supplementary reading material by students of Primary teaching Faculty. In this score book students will get interesting and essential information relate to elementary mathematics. All of topics are cozy to understand and clearable. Furthermore all topics concern to syllabus of subject. Mathematical rigor and clarity often bite each other. At some places, not all subtleties are fully presented. In regards this book will aid students be informed about the main parts of elementary mathematics. We hope that this book will be math guide for bachelors.

AUTHORS

Content

§1. Concept of sets, types of sets: finite and infinite sets. Methods of granting majority. A subset. Universal set. Numerical sets and their representation on the coordinate axis. Union and intersection properties of sets, illustration with Euler-Venn diagrams

In Mathematics, sets are defined as the collection of objects whose elements are fixed and can not be changed. In other words, a set is well defined as the collection of data that does not carry from person to person. The elements can not be repeated in the set but can be written in any order. The set is represented by capital letters.

The empty set, finite set, equivalent set, subset, universal set, superset, and infinite set are some types of set. Each type of set has its own importance during calculations. Basically, in our day-to-day life, sets are used to represent bulk data and collection of data. So, here in this article, we are going to learn and discuss the universal set.

What is Set, What are Types of Sets, and Their Symbols?

A set is well defined as the collection of data that does not carry from person to person.

1. Empty Sets

The set, which has no elements, is also called a null set or void set. It is denoted by {}.

Below are the two examples of the empty set.

Example of empty set: Let set A = {a: a is the number of students studying in Class 6th and Class 7th}. As we all know, a student cannot learn in two classes, therefore set A is an empty set.

Another example of an empty set is set B = {a: 1 < a < 2, a is a natural number}, we know a natural number cannot be a decimal, therefore set B is a null set or empty set.

2. Singleton Sets

The set which has just one element is named a singleton set.

For example,Set A = { 8 } is a singleton set.

3. Finite and Infinite Sets

A set that has a finite number of elements is known as a finite set, whereas the set whose elements can't be estimated, but has some figure or number, which is large to precise in a set, is known as infinite set.

For example, set A = {3,4,5,6,7} is a finite set, as it has a finite number of elements.

Set C = {number of cows in India} is an infinite set, there is an approximate number of cows in India, but the actual number of cows cannot be expressed, as the numbers could be very large and counting all cows is not possible.

4. Equal Sets

If every element of set A is also the elements of set B and if every element of set B is also the elements of set A, then sets A and B are called equal sets. It means set A and set B have equivalent elements and that we can denote it as:

A = B

For example, let

$$A = \{3,4,5,6\}$$

and

$$B = \{6,5,4,3\},$$

then

$$A = B$$

And if

A = {set of even numbers} and

B = { set of natural numbers}

then

$$A \neq B,$$

because natural numbers consist of all the positive integers starting from 1, 2, 3, 4, 5 to infinity, but even numbers start with 2, 4, 6, 8, and so on.

5. Subsets

A set S is said to be a subset of set T if the elements of set S belong to set T, or you can say each element of set S is present in set T. Subset of a set is denoted by the symbol (⊂) and written as

$$S \subset T.$$

We can also write the subset notation as:

$$S \subset T$$

if

$$p \in S \Rightarrow p \in T$$

According to the equation given above, “S is a subset of T only if ‘p’ is an element of S as well as an element of T.” Each set is a subset of its own set, and a void set or empty set is a subset of all sets.

6. Power Sets

The set of all subsets is known as power sets. We know the empty set is a subset of all sets, and each set is a subset of itself. Taking an example of set X = {2,3}. From the above-given statements, we can write,

{∅} is a subset of {2,3}

{2} is a subset of {2,3}

{3} is a subset of {2,3}

{2,3} is also a subset of {2,3}

Therefore, power set of

X = {2,3},

P(X) = {{∅},{2},{3},{2,3}}

7. Universal Sets

A set that contains all the elements of other sets is called a universal set. Generally, it is represented as ‘U.’

For example, set

$$A = \{1,2,3\},$$

set

$$B = \{3,4,5,6\},$$

and

$$C = \{5,6,7,8,9\}.$$

Then, we will write the universal set as,

$$U = \{1,2,3,4,5,6,7,8,9,\}.$$

Note: According to the definition of the universal set, we can say that all the sets are subsets of the universal set.

Therefore,

$$A \subset U$$

$$B \subset U$$

and

$$C \subset U$$

8. Disjoint Sets

If two sets X and Y do not have any common elements, and their intersection results in zero (0), then set X and Y are called disjoint sets. It can be represented as:

$$X \cap Y = \{0\}.$$

Union, Intersection, Difference, and Complement of Sets

1. Union of Sets

The union of two sets consists of all their elements. It is denoted by (⋃).

For example, set A = {2,3,7} and set B = { 4,5,8}.

Then the union of set A and set B will be:

A ⋃ B = {2,3,7,4,5,8}

(Image will be Uploaded Soon)

2. Intersection of Sets

The set of all elements, which are common to all the given sets, gives an intersection of sets. It is denoted by ⋂.

For example, set A = {2,3,7} and set B = {2,4,9}.

So, A ⋂ B = {2}

(Image will be Uploaded Soon)

3. Difference of Sets

The difference between set S and set T is such that it has only those elements which are in the set S and not in the set T.

S – T = {p : p ∊ S and p ∉ T}

Similarly, T – S = {p: p ∊ T and p ∉ S}.

4. Complement of a Set

Let U be the universal set and let A ⊂ U. Then, the complement of A, denoted by A’ or (U - A), is defined as

A’ = {x   U : x A}

X A’ x A

(Image will be Uploaded Soon)

Every set has a complement of sets. Also, for a universal set, the empty set is known as the complement of the universal set. The empty set contains no elements of the subset and is also known as null set, which is denoted by {Ø} or {}.

Questions to be Solved

Question 1. If set A = {a, b, c, d} and B = {b, c, e, f} then, find A-B.

Answer: Let’s find the difference of the two sets,

A – B = {a, d}

and

B – A = {e, f}

Question 2. Let X = {David, Jhon, Misha} be the set of students of Class XI, who are in the school hockey team. Let Y = {Zoya, Rahul, Riya} be the set of students from Class XI who are in the school football team. Find X U Y and interpret the set.

Answer: (U union – combination of two sets)

Given

X = {David, Jhon, Zoya}

Y = {Zoya, Rahul, Riya}

Common elements (Zoya) should be taken once

X U Y = {David, Jhon, Zoya, Rahul, Riya}.

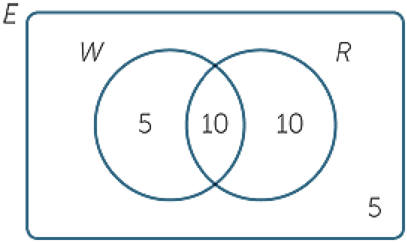
This union set is equal to the set of students from Class XI who are present in the hockey team or the football team or both of the teams.

In all sorts of situations we classify objects into sets of similar objects and count them. This procedure is the most basic motivation for learning the whole numbers and learning how to add and subtract them.

Such counting quickly throws up situations that may at first seem contradictory.

'Last June, there were 15 windy days and 20 rainy days, yet 5 days were neither windy nor rainy.'

How can this be, when June only has 30 days? A Venn diagram, and the language of sets, easily sorts this out.



Let W be the set of windy days,and R be the set of rainy days.Let E be the set of days in June.Then W and R; together have size 25, sothe overlap between W and R is 10.; The Venn diagram opposite displays; the whole situation.

The purpose of this module is to introduce language for talking about sets, and some notation for setting out calculations, so that counting problems such as this can be sorted out. The Venn diagram makes the situation easy to visualise.

A set is just a collection of objects, but we need some new words and symbols and diagrams to be able to talk sensibly about sets.

In our ordinary language, we try to make sense of the world we live in by classifying collections of things. English has many words for such collections. For example, we speak of ‘a flock of birds’, ‘a herd of cattle’, ‘a swarm of bees’ and ‘a colony of ants’.

We do a similar thing in mathematics, and classify numbers, geometrical figures and other things into collections that we call sets. The objects in these **sets** are called the **elements** of the set.

Describing a set

A set can be described by **listing** all of its elements. For example,

$$S = \{ 1, 3, 5, 7, 9 \},$$

which we read as ‘S is the set whose elements are 1, 3, 5, 7 and 9’. The five elements of the set are separated by commas, and the list is enclosed between curly brackets.

A set can also be described by **writing a description** of its elements between curly brackets. Thus the set S above can also be written as

$$S = \{ \text{odd whole numbers less than 10} \},$$

which we read as ‘S is the set of odd whole numbers less than 10’.

A set must be **well defined**. This means that our description of the elements of a set is clear and unambiguous. For example, { tall people } is not a set, because people tend to disagree about what 'tall' means. An example of a well-defined set is

T = { letters in the English alphabet }.

Equal sets

Two sets are called **equal** if they have exactly the same elements. Thus following the usual convention that 'y' is not a vowel,

{ vowels in the English alphabet } = { a, e, i, o, u }

On the other hand, the sets { 1, 3, 5 } and { 1, 2, 3 } are not equal, because they have different elements. This is written as

$\{ 1, 3, 5 \} \neq \{ 1, 2, 3 \}$.

The order in which the elements are written between the curly brackets does not matter at all. For example,

{ 1, 3, 5, 7, 9 } = { 3, 9, 7, 5, 1 } = { 5, 9, 1, 3, 7 }.

If an element is listed more than once, it is only counted once. For example,

{ a, a, b } = { a, b }.

The set { a, a, b } has only the two elements a and b. The second mention of a is an unnecessary repetition and can be ignored. It is normally considered poor notation to list an element more than once.

The symbols ∈ and ∉

The phrases 'is an element of' and 'is not an element of' occur so often in discussing sets that the special symbols ∈ and ∉ are used for them. For example, if

A = { 3, 4, 5, 6 },

then

$3 \in A$ (Read this as '3 is an element of the set A'.)

8 ∉ A (Read this as ‘8 is not an element of the set A’.)

Describing and naming sets

- A **set** is a collection of objects, called the **elements** of the set.
- A set must be **well defined**, meaning that its elements can be described and listed without ambiguity. For example:

{ 1, 3, 5 } and { letters of the English alphabet }.

- Two sets are called **equal** if they have exactly the same elements.
  - ○ The order is irrelevant.
  - ○ Any repetition of an element is ignored.
  - ○ If a is an element of a set S, we write a ∈ S.
  - ○ If b is not an element of a set S, we write b ∉ S.

EXERCISE 1

a) Specify the set A by listing its elements, where A = { whole numbers less than 100 divisible by 16 }.

b) Specify the set B by giving a written description of its elements, where

B = { 0, 1, 4, 9, 16, 25 }.

c) Does the following sentence specify a set? C = { whole numbers close to 50 }.

Finite and infinite sets

All the sets we have seen so far have been finite sets, meaning that we can list all their elements. Here are two more examples:

{ whole numbers between 2000 and 2005 } = { 2001, 2002, 2003, 2004 }

{ whole numbers between 2000 and 3000 } = { 2001, 2002, 2003,…, 2999 }

The three dots '…' in the second example stand for the other 995 numbers in the set. We could have listed them all, but to save space we have used dots instead. This notation can only be used if it is completely clear what it means, as in this situation.

A set can also be **infinite** − all that matters is that it is well defined. Here are two examples of infinite sets:

{ even whole numbers } = { 0, 2, 4, 6, 8, 10, …}

{ whole numbers greater than 2000 } = { 2001, 2002, 2003, 2004, …}

Both these sets are infinite because no matter how many elements we list, there are always more elements in the set that are not on our list. This time the dots '…' have a slightly different meaning, because they stand for infinitely many elements that we could not possibly list, no matter how long we tried.

The numbers of elements of a set

If S is a finite set, the symbol | S | stands for the **number of elements** of S. For example:

If S = { 1, 3, 5, 7, 9 }, then | S | = 5.

If

A = { 1001, 1002, 1003, …, 3000 },

then

| A | = 2000.

If

T = { letters in the English alphabet },

then

| T | = 26.

The set

S = { 5 }

is a **one-element set** because

| S | = 1.

It is important to distinguish between the number 5 and the set S = { 5 }:

$$5 \in S$$

but

$$5 \neq S\ .$$

The empty set

The symbol ∅ represents the empty set, which is the set that has no elements at all. Nothing in the whole universe is an element of ∅:

$$|\,\emptyset\,| = 0 \text{ and } x \notin \emptyset, \text{ no matter what } x \text{ may be.}$$

There is only one empty set, because any two empty sets have exactly the same elements, so they must be equal to one another.

Finite and Infinite sets

- A set is called **finite** if we can list all of its elements.
- An **infinite** set has the property that no matter how many elements we list,there are always more elements in the set that are not on our list.
- If S is a finite set, the symbol | S | stands for the number of elements of S.
- The set with no elements is called the empty set, and is written as ∅. Thus | ∅ | = 0.
- A one-element set is a set such as S = { 5 } with | S | = 1.

EXERCISE 2

a) Use dots to help list each set, and state whether it is finite or infinite.

I. B = { even numbers between 10 000 and 20 000 }

II. A = { whole numbers that are multiples of 3 }

b) If the set S in each part is finite, write down | S |.

I. S = { primes }

II. S = { even primes }

III. S = { even primes greater than 5 }

IV. S = { whole numbers less than 100 }

c) Let F be the set of fractions in simplest form between 0 and 1 that can be written with a single-digit denominator. Find F and | F |.

SUBSETS AND VENN DIAGRAMS

Subsets of a set

Sets of things are often further subdivided. For example, owls are a particular type of bird, so every owl is also a bird. We express this in the language of sets by saying that the set of owls is a subset of the set of birds.

A set S is called a subset of another set T if every element of S is an element of T. This is written as

S ⊆ T (Read this as ‘S is a subset of T’.)

The new symbol ⊆ means ‘is a subset of’. Thus

{ owls } ⊆ { birds }

because every owl is a bird. Similarly,

İf

A = { 2, 4, 6 }

and

B = { 0, 1, 2, 3, 4, 5, 6 },

Then

A ⊆ B,

because every element of A is an element of B.

The sentence ‘S is not a subset of T’ is written as

S ⊈ T.

This means that at least one element of S is not an element of T. For example,

{ birds } ⊈ { flying creatures }

because an ostrich is a bird, but it does not fly. Similarly, if

$$A = \{ 0, 1, 2, 3, 4 \}$$

and

$$B = \{ 2, 3, 4, 5, 6 \},$$

then

$$A \not\subseteq B,$$

because $0 \in A$, but $0 \notin B$.

The set itself and the empty set are always subsets

Any set S is a subset of itself, because every element of S is an element of S. For example:

$$\{ \text{birds} \} \subseteq \{ \text{birds} \}$$

and

$$\{ 1, 2, 3, 4, 5, 6 \} = \{ 1, 2, 3, 4, 5, 6 \}.$$

Furthermore, the empty set $\emptyset$ is a subset of every set S, because every element of the empty set is an element of S, there being no elements in $\emptyset$ at all. For example:

$$\emptyset \subseteq \{ \text{birds} \}$$

and

$$\emptyset \subseteq \{ 1, 2, 3, 4, 5, 6 \}.$$

Every element of the empty set is a bird, and every element of the empty set is one of the numbers 1, 2, 3, 4, 5 or 6.

Subsets and the words ‘all’ and ‘if … then’

A statement about subsets can be rewritten as a sentence using the word ‘all’.For example, ‘All owls are birds.’

{owls } ⊆ { birds }	means	
{ multiples of 4 } ⊆ { even numbers }	means	‘All multiples of 4 are even.’

{ rectangles } ⊆ { rhombuses }	means	‘Not all rectangles are rhombuses.’

They can also be rewritten using the words ‘if … then’. For example,

{ owls } ⊆ { birds }	means	‘If a creature is an owl, then it is a bird.’
{multiples of 4 } ⊆ { even numbers }	means	‘If a number is a multiple of 4, then it is even’:
{rectangles } ⊆ { rhombuses }	means	‘If a figure is a rectangle, then it may not be a square.’

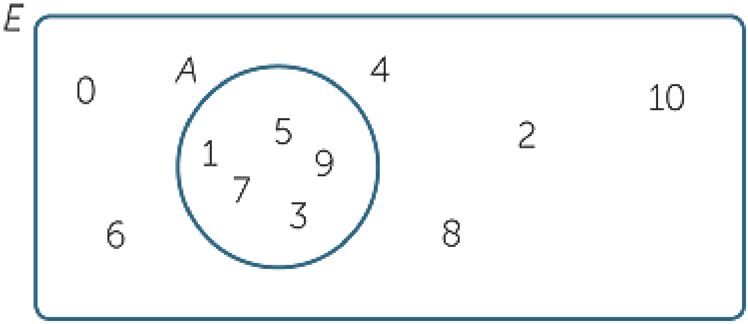
Venn diagrams

Diagrams make mathematics easier because they help us to see the whole situation at a glance. The English mathematician John Venn (1834−1923) began using diagrams to represent sets. His diagrams are now called **Venn diagrams**.

In most problems involving sets, it is convenient to choose a larger set that contains all of the elements in all of the sets being considered. This larger set is called the **universal set**, and is usually given the symbol E. In a Venn diagram, the universal set is generally drawn as a large rectangle, and then other sets are represented by circles within this rectangle.

For example, if V = { vowels }, we could choose the universal set as E = { letters of the alphabet } and all the letters of the alphabet would then need to be placed somewhere within the rectangle, as shown below.

In the Venn diagram below, the universal set is E = { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 }, and each of these numbers has been placed somewhere within the rectangle.

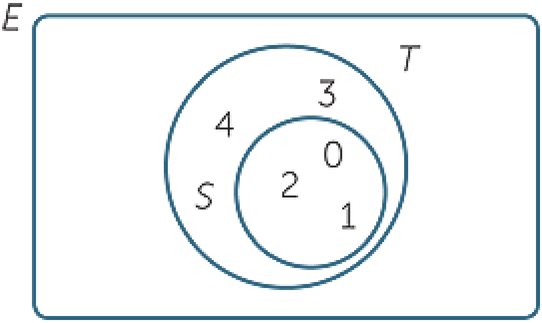


The region inside the circle represents the set A of odd whole numbers between 0 and 10. Thus we place the numbers 1, 3, 5, 7 and 9 inside the circle, because A = { 1, 3, 5, 7, 9 }. Outside the circle we place the other numbers 0, 2, 4, 6, 8 and 10 that are in E but not in A.

Representing subsets on a Venn diagram

When we know that S is a subset of T, we place the circle representing S inside the circle representing T. For example, let S =

{ 0, 1, 2 }, and T = { 0, 1, 2, 3, 4 }. Then S is a subset of T, as illustrated in the Venn diagram below.

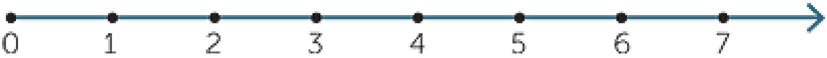


Make sure that 5, 6, 7, 8, 9 and 10 are placed outside both circles>

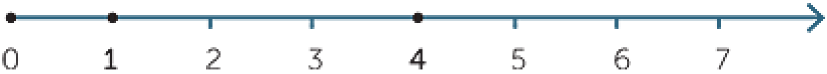
Subsets and the number line

The whole numbers are the numbers 0, 1, 2, 3,… These are often called the 'counting numbers', because they are the numbers we use when counting things. In particular, we have been using these numbers to count the number of elements of finite sets. The number zero is the number of elements of the empty set.

The set of all whole numbers can be represented by dots on the number line.



Any finite subset of set of whole numbers can be represented on the number line. For example, here is the set { 0, 1, 4 }.



Subsets of a st

- If all the elements of a set S are elements of another set T, then S is called a subset of T. This is written as S ⊆ T.

- If at least one element of S is not an element of T, then S is not a subset of T. This is written as S $\not\subseteq$ T.
- If S is any set, then ∅ ⊆ S and S ⊆ S.
- A statement about a subset can be rewritten using the words ‘all’ or ‘if … then’.
- Subsets can be represented using a **Venn diagram**.
- The set { 0, 1, 2, 3, 4, … } of whole numbers is infinite.
- The set of whole numbers, and any finite subset of them, can be represented on the number line.

EXERCISE 3

a) Rewrite in set notation:

I. All squares are rectangles.

II. Not all rectangles are rhombuses.

b) Rewrite in an English sentence using the words ‘all’ or ‘not all’:

I. { whole number multiples of 6 } ⊆ { even whole numbers }.

II.{ square whole numbers } ⊆ { even whole numbers }.

c) Rewrite the statements in part (b) in an English sentence using the words ‘if …, then’.

d) Given the sets

A = { 0, 1, 4, 5 }

And

B = { 1, 4 }:

I. Draw a Venn diagram of A and B using the universal set

U = { 0, 1, 2, … , 8 }.

II. Graph A on the number line.

Suppose that a suitable universal set E has been chosen. The **complement** of a set Sis the set of all elements of E that are not in S. The complement of S is written as S

For example,

If

$$E = \{ \text{letters} \}$$

and

$$V = \{ \text{vowels} \},$$

then

$$V^c = \{ \text{consonants} \}$$

If

$$E = \{ \text{whole numbers} \}$$

and

$$O = \{ \text{odd whole numbers} \},$$

then

$$O^c = \{\text{even whole numbers}\}.$$

Complement and the word ‘not’

The word ‘not’ corresponds to the complement of a set. For example, in the two examples above,

$V^c = \{$ letters that are not vowels $\} = \{$ consonants $\}$

$O^c = \{$ whole numbers that are not odd $\} = \{$ even whole numbers $\}$

The set V^c in the first example can be represented on a Venn diagram as follows.

§2. The intersection of two sets

The intersection of two sets A and B consists of all elements belonging to A and to B.This is written as A ∩ B. For example, some musicians are singers and some play an instrument.

If A = { singers } and B = { instrumentalists }, then

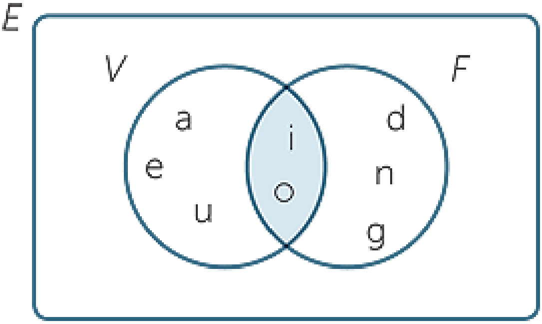
A ∩ B = { singers who play an instrument }.

Here is an example using letters.

If V = { vowels } and F = { letters in 'dingo' }, then

V ∪ F = { i, o }.

This last example can be represented on a Venn diagram as follows.



Intersection and the word 'and'

The word 'and' tells us that there is an intersection of two sets. For example:

{ singers } ∩ { instrumentalists } = { people who sing and play an instrument }

{ vowels } ∩ { letters of 'dingo' } = { letters that are vowels and are in 'dingo' }

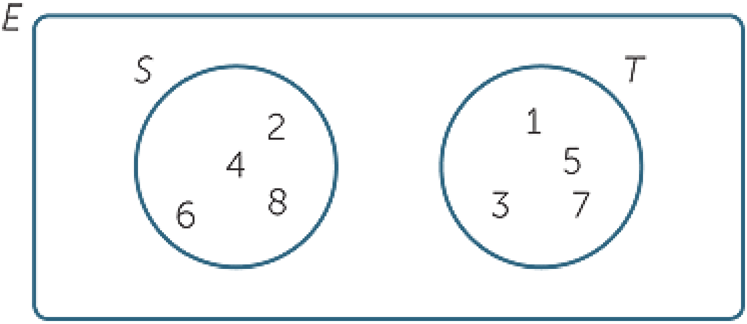
The union of two sets

The union of two sets A and B consists of all elements belonging to A or to B. This is written as A ∪ B. Elements belonging to both set belong to the union. Continuing with the example of singers and instrumentalists:

If A = { singers } and B = { instrumentalists }, then A ∪ B = { musical performers }.

In the case of the sets of letters:

If V = { vowels } and F = { letters in ‘dingo’ }, then V &acup; F = { a, e, i, o, u, d, n, g }.



Union and the word ‘or’

The word ‘or’ tells us that there is a union of two sets. For example:

{ singers } ∪ { instrumentalists } = { people who sing or play an instrument }

{ vowels } ∪ { letters in ‘dingo’ } = { letters that are vowels or are in ‘dingo’ }

The word ‘or’ in mathematics always means ‘and/or’, so there is no need to add ‘or both’ to these descriptions of the unions. For example,

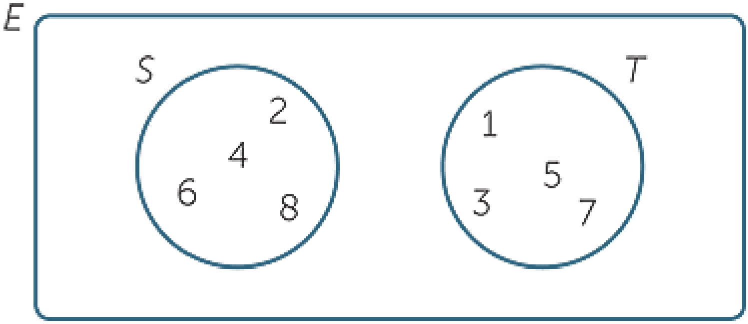
A = { 0, 2, 4, 6, 8, 10, 12, 14 }
and
B = { 0, 3, 6, 9, 12 },
then
A ∪ B = { 0, 2, 3, 4, 6, 8, 9, 10, 12, 14 }.

Here the elements 6 and 12 are in both sets A and B.

Disjoint sets

Two sets are called disjoint if they have no elements in common. For example:

The sets S = { 2, 4, 6, 8 } and T = { 1, 3, 5, 7 } are disjoint.



Another way to define disjoint sets is to say that their intersection is the empty set,

Two sets A and B are disjoint if $A \cap B = \emptyset$.

In the example above,

$S \cap T = \emptyset$ because no number lies in both sets.

Complement, intersection and union

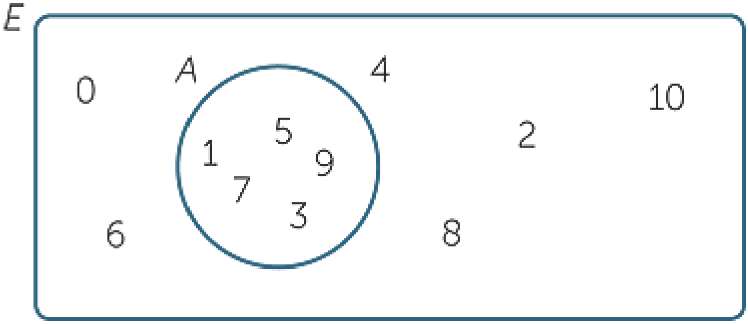
Let A and B be subsets of a suitable universal set E.

- The complement A^c is the set of all elements of E that are not in A.
- The intersection $A \cap B$ is the set of all elements belonging to A and to B.
- The union $A \cup B$ is the set of all elements belonging to A or to B.
- In mathematics, the word ‘or’ always means ‘and/or’, so all the elements that are in both sets are in the union.

- The sets A and B are called disjoint if they have no elements in common, that is,if A ∩ B = ∅.

Representing the complement on a Venn diagram

Let A = { 1, 3, 5, 7, 9 } be the set of odd whole numbers less than 10, and take the universal set as E = { 0, 1, 2, … , 10 }. Here is the Venn diagram of the situation.



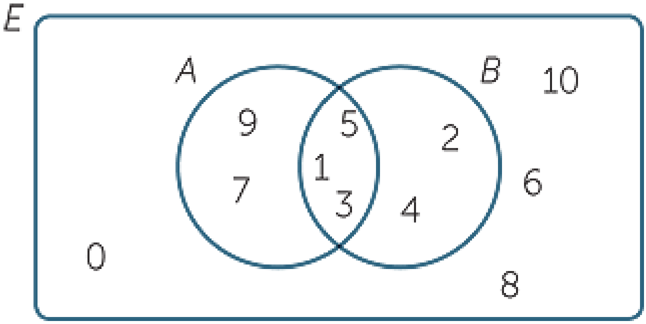
The region inside the circle represents the set A, so we place the numbers 1, 3, 5, 7 and 9 inside the circle. Outside the circle, we place the other numbers 0, 2, 4, 6, 8 and 10 that are not in A. Thus the region outside the circle represents the complement A^c = {0, 2, 4, 6, 8, 10}.

Representing the intersection and union on a Venn diagram

The Venn diagram below shows the two sets

A = { 1, 3, 5, 7, 9 } and B = { 1, 2, 3, 4, 5 }.

- The numbers 1, 3 and 5 lie in both sets, so we place them in the overlapping region of the two circles.
- The remaining numbers in A are 7 and 9. These are placed inside A, but outside B.
- The remaining numbers in B are 2 and 4. These are placed inside B, but outside A.



Thus the overlapping region represents the intersection A ∩ B = { 1, 3, 5 }, and the two circles together represent the union A ∪ B = { 1, 2, 3, 4, 5, 7, 9 }.

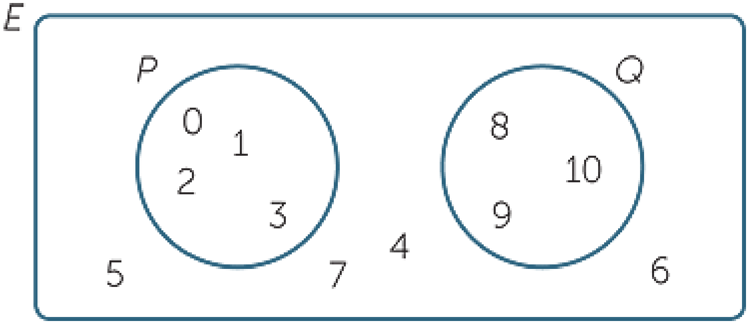
The four remaining numbers 0, 6, 8 and 10 are placed outside both circles.

Representing disjoint sets on a Venn diagram

When we know that two sets are disjoint, we represent them by circles that do not intersect. For example, let

P = { 0, 1, 2, 3 } and Q = { 8, 9, 10 }

Then P and Q are disjoint, as illustrated in the Venn diagram below.



Venn diagrams with complements, unions and intersections

- Sets are represented in a Venn diagram by circles drawn inside a rectangle representing the universal set.

- The region outside the circle represents the complement of the set.
- The overlapping region of two circles represents the intersection of the two sets.
- Two circles together represent the union of the two sets.
- When two sets are disjoint, we can draw the two circles without any overlap.
- When one set is a subset of another, we can draw its circle inside the circle of the other set.

EXERCISE 4

Let the universal set be E = {whole numbers less than 20 }, and let

A = { squares less than 20 }
B = { even numbers less than 20 }
C = { odd squares less than 20 }

a) Draw A and C on a Venn diagram, and place the numbers in the correct regions.

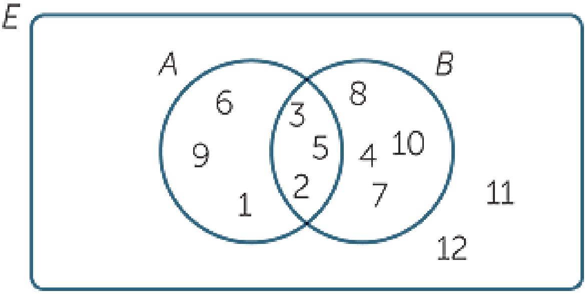
b) Draw B and C on a Venn diagram, and place the numbers in the correct regions.

c) Shade A ∩ B on a Venn diagram, and place the numbers in the correct regions.

d) Shade A ∪ B on a Venn diagram, and place the numbers in the correct regions.

SOLVING PROBLEMS USING A VENN DIAGRAM

Before solving problems with Venn diagrams, we need to work out how to keep count of the elements of overlapping sets.



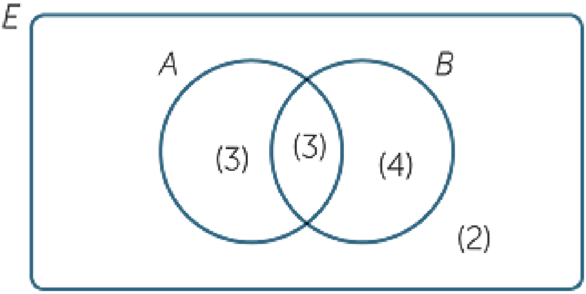
The upper diagram to the right shows two sets A and B inside a universal set E, where

| A | = 6

And

| B | = 7,

with 3 elements in the intersection A ∩ B.



The lower diagram to the right shows only the number of elements in each of the four regions.

These numbers are placed inside round brackets so that they don’t look like elements.

You can see from the diagrams that

| A | = 6

And

| B | = 7,

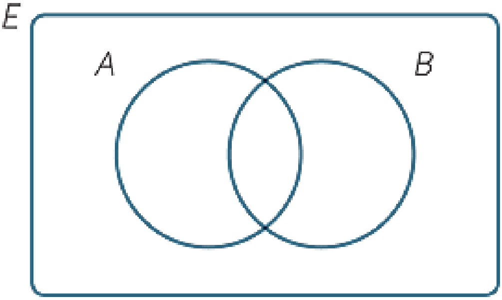
But

$$|A \cup B| \neq 6 + 7.$$

The reason for this is that the elements inside the overlapping region $A \cap B$ should only be counted once, not twice. When we subtract the three elements of $A \cap B$ from the total, the calculation is then correct.

$$|A \cup B| = 6 + 7 - 3 = 10.$$

EXAMPLE



In the diagram to the right,

$$|A| = 15,$$
$$|B| = 25,$$
$$|A \cap B| = 5$$

And

$$|E| = 50.$$

a) Insert the number of elements into each of the four regions.

b) Hence find $|A \cup B|$ and $|A \cap B^c|$

SOLUTION

a) We begin at the intersection and work outwards.

The intersection $A \cap B$ has 5 elements.

Hence the region of A outside $A \cap B$ has 10 elements, and the region of B outside $A \cap B$ has 20 elements. This makes 35 elements so far, so the outer region has 15 elements.

b

From the diagram,

$$| A \cup B | = 35$$

And

$$| A \cap B^c | = 10.$$

EXERCISE 5

a Draw a Venn diagram of two sets S and T

b Given that $| S | = 15$, $| T | = 20$, $| S \cup T | = 25$ and $| E | = 50$, insert the number of elements into each of the four regions.

c Hence find $| S \cap T |$ and $| S \cap T^c |$.

Number of elements in the regions of a Venn diagram

The numbers of elements in the regions of a Venn diagram can bed one by working systematically around the diagram.

The number of elements in the union of two sets A and B is

Number of elements in A ∪ B = number of elements in A

Number of elements in A ∪ B = number of elements in A

+ number of elements in B

− number of elements in A ∩ B.

Writing this formula in symbols,

$$| A \cup B | = | A | + | B | - | A \cap B |.$$

Solving problems by drawing a Venn diagram

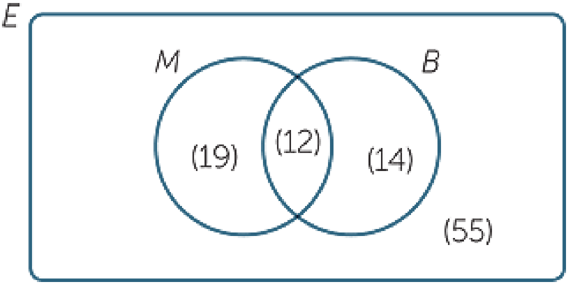
Many counting problems can be solved by identifying the sets involved, then drawing up a Venn diagram to keep track of the numbers in the different regions of the diagram.

EXAMPLE

A travel agent surveyed 100 people to find out how many of them had visited the cities of Melbourne and Brisbane. Thirty-one people had visited Melbourne, 26 people had been to Brisbane, and 12 people had visited both cities. Draw a Venn diagram to find the number of people who had visited:

- **a** Melbourne or Brisbane
- **b** Brisbane but not Melbourne
- **c** only one of the two cities
- **d** neither city.

SOLUTION



Let M be the set of people who had visited Melbourne, and let B be the set of people who had visited Brisbane. Let the universal set E be the set of people surveyed.

The information given in the question can now be rewritten as

$$|M| = 31,$$
$$|B| = 26,$$
$$|M \cap B| = 12$$

and

$$|E| = 100.$$

Hence number in M only $= 31 - 12$

$= 19$

and number in B only $= 26 - 12$

$= 14.$

a Number visiting Melbourne or Brisbane = 19 + 14 +12 = 45.

b Number visiting Brisbane only = 14.

c Number visiting only one city = 19 + 14 = 33.

d Number visiting neither city = 100 − 45 = 55.

Problem solving using Venn diagrams

- First identify the sets involved.
- Then construct a Venn diagram to keep track of the numbers in the different regions of the diagram.

EXERCISE 6

Twenty-four people go on holidays. If 15 go swimming, 12 go fishing, and 6 do neither, how many go swimming and fishing? Draw a Venn diagram and fill in the number of people in all four regions.

EXERCISE 7

In a certain school, there are 180 pupils in Year 7. One hundred and ten pupils study French, 88 study German and 65 study Indonesian. Forty pupils study both French and German, 38 study German and German only. Find the number of pupils who study:

all three languages

none of the languages

Indonesian only

at least one language

either one ot two of the three languages.

The examples in this module have shown how useful sets and Venn diagrams are in counting problems. Such problems will continue to present themselves throughout secondary school.

The language of sets is also useful for understanding the relationships between objects of different types. For example, we have met various sorts of numbers, and we can summarise some of our knowledge very concisely by writing

{ whole numbers } ⊆ { integers } ⊆ { rational numbers } ⊆ { real numbers }.

The relationships amongst types of special quadrilaterals is more complicated. Here are some statements about them.

{ squares } ⊆ { rectangles } ⊆ { parallelograms } ⊆ { trapezia }

{ rectangles } ∩ { rhombuses } = { squares }

If A = { convex kites } and B = { non-convex kites }, then

A ∩ B = ∅ and A ∪ B = { kites }

That is, the set of convex kites and the set of non-convex kites are disjoint, but their union is the set of all kites.

It is far easier to talk about probability using the language of sets. The set of all outcomes is called the sample space, a subset of the sample space is called an event. Thus when we throw three coins, we can take the sample space as the set

S = { HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

and the event ‘throwing at least one head and at least one tail’ is then the subset

E = { HHT, HTH, HTT, THH, THT, TTH }

Since each outcome is equally likely,

P(at least one head and at least one tail) = $\frac{|E|}{|S|} = \frac{3}{4}$.

The event space of the complementary event ‘throwing all heads or all tails’ is the complement of the event space in the sample space, which we take as the universal set, so

E^c = { HHH, TTT }.

Since | E | + | E^c | = | S |, it follows after dividing by | S | that P(E^c) = 1 − P(E), so

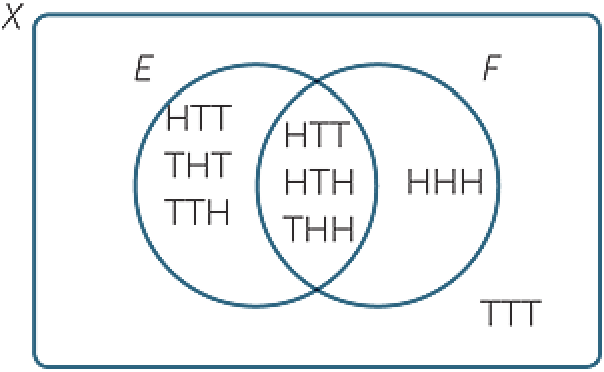
P(throwing all head or all tails) = $1 - \frac{3}{4} = \frac{1}{4}$.

Let F be the event 'throwing at least two heads'. Then

F = { HHH, HHT, HTH, THH }

A Venn diagram is the best way to sort out the relationship between the two events E and F. We can then conclude that

P(E and F) = 3 and P(E or F) = 7



Sets and Functions

When we discuss a function, we usually want to write down its domain − the set of all x-values that we can substitute into it, and its range − the set of all y-values that result from such substitutions.

For example, for the function $y = x^2$,

domain = { real numbers } and range = {y: y ≥ 0}.

The notation used here for the range is 'set-builder notation', which is no longer taught in school. Consequently we mostly avoid set notation altogether, and use instead less rigorous language, ‘The domain is all real numbers, and the range is $y > 0$.’

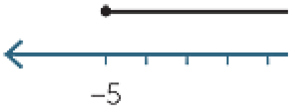
Speaking about the condition rather than about the set, however, can confuse some students, and it is often useful to demonstrate the set theory ideas lying behind the abbreviated notation.

Sets and equations

Here are two inequalities involving absolute value and their solution.

$|x| \le 5$

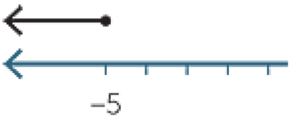
(distance from x to 0) ≤ 5



$x \ge -5$ and $x \le 5$.

$|x| \ge 5$

(distance from x to 0) ≥ 5



$x \le -5$ and $x \ge 5$.

If we use the language of solution sets, and pay attention to ‘and’ and ‘or’, we see that the solution of the first inequality is the intersection of two sets, and the solution of the second inequality is the union of two sets. In set-builder notation, the solutions to the two inequalities are

$$\{ x: x \ge -5 \} \cap \{ x: x \le 5 \} = \{ x: -5 \le x \le 5\}, \text{ and}$$

$$\{ x: x \le -5 \} \cap \{ x: x \ge 5 \} = \{ x: x \text{ £ } -5 \text{ or } x \ge 5\}.$$

At school, however, we simply write the solutions to the two inequalities as the conditions alone,

$$-5 \le x \le 5 \qquad \text{and} \qquad x \le -5 \text{ or } x \ge 5$$

There are many similar situations where the more precise language of sets may help to clarify the solutions of equations and inequalities when difficulties are raised during discussions.

Counting problems go back to ancient times. Questions about 'infinity' were also keenly discussed by mathematicians in the ancient world. The idea of developing a 'theory of sets', however, only began with publications of the German mathematician Georg Cantor in the 1870s, who was encouraged in his work by Karl Weierstrass and Richard Dedekind, two of the greatest mathematicians of all time.

Cantor's work involved the astonishing insight that there are infinitely many different types of infinity. In the hierarchy of infinities that he discovered, the infinity of the whole numbers is the smallest type of infinity, and is the same as the infinity of the integers and of the rational numbers. He was able to prove, quite simply, that the infinity of the real numbers is very much larger, and that the infinity of functions is much larger again. His work caused a sensation and some Catholic theologians criticised his work as jeopardising 'God's exclusive claim to supreme infinity'.

Cantor's results about types of infinity are spectacular and not particularly difficult. The topic is quite suitable as extension work at school, and the basic ideas have been presented in some details in Appendix 2 of the Module **The Real Numbers.**

Cantor's original version of set theory is now regarded as 'naive set theory', and contains contradictions. The most famous of these contradictions is called 'Russell's paradox', after the British philosopher and mathematician Bertrand Russell. It is a version of the ancient barber-paradox,

'A barber shaves all those who do not shave themselves. Who shaves the barber?'

and it works like this:

> 'Sets that are members of themselves are rather unwelcome objects.

> In order to distinguish such tricky sets from the ordinary, well-behaved sets,
> let S be the set of all sets that are not members of themselves.
> But when we consider the set S itself, we have a problem.
> If S is a member of S, then S is not a member of S.
> If S is not a member of S, then S is a member of S.
> This is a contradiction.'

The best-known response, but by no means the only response, to this problem and to the other difficulties of 'naive set theory' is an alternative, extremely sophisticated, formulation of set theory called 'Zermelo-Fraenkel set theory', but it is hardly the perfect solution. While no contradictions have been found,many disturbing theorems have been proven. Most famously, Kurt Goedel proved in 1931 that it is impossible to prove that Zermelo-Fraenkel set theory, and indeed any system of axioms within which the whole numbers can be constructed, does not contain a contradiction!

Nevertheless, set theory is now taken as the absolute rock-bottom foundation of mathematics, and every other mathematical idea is defined in terms of set theory. Thus despite the paradoxes of set theory, all concepts in geometry, arithmetic, algebra and calculus − and every other branch of modern mathematics − are defined in terms of sets, and have their logical basis in set theory.

ANSWERS TO EXERCISES

EXERCISE 1

a A = { 0, 16, 32, 48, 64, 80, 96 }.

b The most obvious answer is B = { square numbers less than 30 }.

c No, because I don't know precisely enough what 'close to' means.

EXERCISE 2

a **I** A = { 10 002, 10 004, … , 19 998 } is finite.
ii B = { 0, 3, 6, … } is infinite.

b	**I**	This set is infinite.	**ii**	| S | = 1.
	Iii	| S | = 0.	**iv**	| S | = 100.

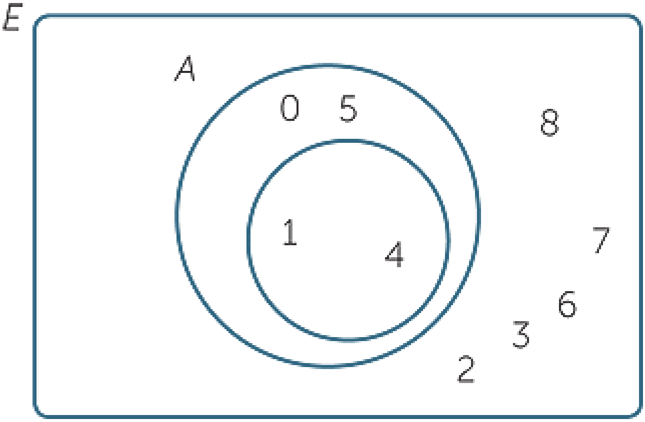
C F = $\left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}\right\}$, so | F | = 27.

EXERCISE 3

A	**i**	{ squares } ⊆ { rectangles }.	**ii**	{ rectangles } ⊆ { rhombuses }.
b	**i**	All multiples of 6 are even.	**ii**	Not all squares are even.

C **i** If a whole number is a multiple of 6, then it is even.

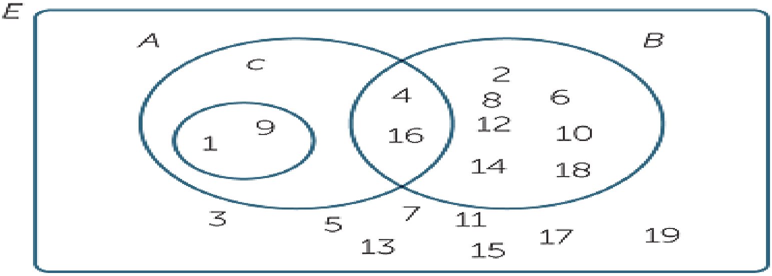
ii If a whole number is a square, then it may not be even.



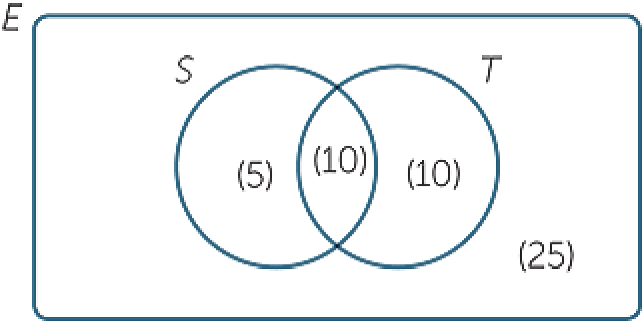
i

EXERCISE 4

a−d



EXERCISE 5



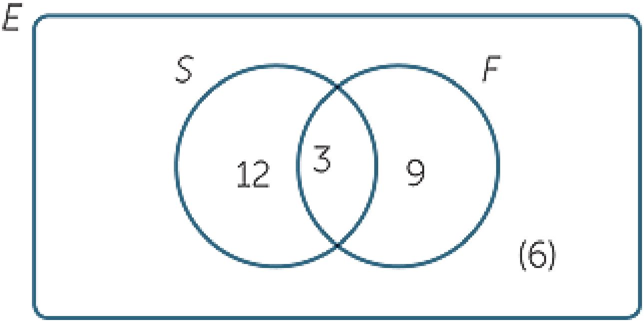
The union S ∪ T has 25 elements, whereas S has 15 elements and T has 20 elements, so the overlap S ∩ T has 10 elements.

Hence the region of S outside S ∩ T has 5 elements, and the region of T outside S ∩ T has 10 elements. Hence the outer region has 50 − 25 = 25 elements.

c From the diagram, | S ∩ T | = 10 and | S ∪ Tc | = 40.

EXERCISE 6

Since only 18 people are involved in swimming or fishing and 15 + 12 = 27, there are 9 people who go swimming and fishing.



EXERCISE 7

a 9 **b** 10 **c** 12 **d** 68 **e** 159

§3. Difference and properties of sets, illustration with Euler-Venn diagram. Subset complement. Separation of sets into pairwise disjoint subsets (classes). Combinatorics problems. Ordering sets. Non-repeating compounds and their types. Unique arrangements, permutations, combinations and their properties

Set definitions, notation, and properties

- Definition

▪ Set; collection of objects

▪ Object in set called element

▪ Elements in a set often share some common property

- Notation

▪ Uppercase letters denote set, e.g., set S

▪ Lowercase letters denote elements, e.g., element k ▪ Braces { } indicate set, e.g., S = {k, s, p} ▪ Symbol ∊ denotes set membership, e.g., k ∊ S, m ∉ S ▪ e.g, A = {violet, chartreuse, burnt umber}, chartreuse ∊ A, magenta ∉ A ▪ e.g, A = {1, 3, 5, 7, …}, 9 ∊ A, 8 ∉ A

Properties of sets

▪ Uniqueness; an element may only occur once in a set, e.g., A = {0, 1} and B = {0, 1, 1} are the same set

▪ Non-order; set elements have no order, e.g., A = {0, 1} and B = {1, 0} are the same set

▪ Equality; sets containing the same elements are equal, e.g., if A = {b, j, s} and B = {j, s, b} then A = B i.e., A = B means $(\forall x)[(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$ ▪ Sets may be finite or infinite (number of elements)

Sets of sets and empty sets

- Sets may be elements of sets ▪ e.g., {0, {1}, {2, {3}}, 4}

• Sets may have no elements, i.e., “empty set” ▪ Written ∅, or sometimes { }, but not { ∅ }

Number sets

• ℕ = “natural numbers”; set of all nonnegative integers (0 ∈ ℕ), i.e., {0, 1, 2, 3, …}, e.g., 86 ∈ ℕ

• ℤ = set of all integers, i.e. {…, -3, -2, -1, 0, 1, 2, 3, …}, e.g., +86 ∈ ℤ • ℚ = set of all rational numbers, e.g., 86/1 ∈ ℚ

• ℝ = set of all real numbers, e.g., 86 π ∈ ℝ

• ℂ = set of all complex numbers, e.g., 86 + 12 i ∈ ℂ

Specifying sets

• Specifying (identifying elements of) sets

▪ Exhaustively list all elements

▪ Partially list elements to show pattern

▪ Use recursion to describe how to generate elements

▪ Describe property P that characterizes elements, informally using words

▪ Describe property P that characterizes elements, formally using equations and/or logic

• Property P

▪ Set whose elements have property P: { x | P (x)} ▪ P (x) is unary predicate ▪ S = { x | P (x)} means (∀x)[(x ∈ S → P (x)) ∧ (P (x) → x ∈ S)]

Examples of specifying sets

From characterizing property to exhaustive list

6a. { x | x ∈ ℕ and x 2 – 5 x + 6 = 0} x 2 – 5 x + 6 = (x – 2)(x – 3) {2, 3}

8a. { x | x ∈ ℕ and (∃ q)(q ∈ {2, 3} and x = 2 q)} {4, 6}

9. A ={2, 4, 8, …} Is 16 ∈ A ? Yes: A = { x | x = 2 n for n a positive integer} No: A = { x | x = 2 + n (n – 1) for n a positive integer} Not enough information

From characterizing property to exhaustive list

$A = \{ x \mid (\exists y)(y \in \{0, 1, 2\} \text{ and } x = y\ 3)\}$

$A = \{ x \mid x \text{ is a cube of } 0, 1, \text{ or } 2\}$

$A = \{0, 1, 8\}$ $B = \{ x \mid x \in \mathbb{N} \text{ and } (\exists y)(y \in \mathbb{N} \text{ and } x \leq y)\}$

$B = \{ x \mid x$ is nonnegative integer $\leq$ some nonnegative integer$\}$ $B = \mathbb{N}$

$C = \{ x \mid x \in \mathbb{N} \text{ and } (\forall y)(y \in \mathbb{N} \rightarrow x \leq y)\}$

$C = \{ x \mid x$ a nonnegative integer $\leq$ all nonnegative integers$\}$ $C = \{0\}$

From exhaustive list to characterizing property

a. $\{1, 4, 9, 16\}$ $\{ x \mid x$ is one of the first four perfect squares$\}$ $\{ x \mid x = y\ 2 \wedge 1 \leq y \leq 4\}$

b. {butcher, baker, candlestick maker} { x | x is one of the “Three Men in a Tub”}

c. $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$ $\{ x \mid x$ is a prime number$\}$ $\{ x \mid ((\exists y)(y \neq 1 \wedge y \neq x \wedge y|x))'\}$

Subsets

- Subset

▪ Set A is a subset of set B iff every element of A is an element of B ▪ e.g., $A = \{2, 3, 5, 12\}$, $B = \{2, 3, 4, 5, 9, 12\}$ A is a subset of B

▪ Notation: $A \subseteq B$ ▪ $A \subseteq B$ iff $(\forall x)(x \in A \rightarrow x \in B)$

- Proper subset

▪ Set A is a proper subset of B iff A is a subset of B and there is at least one element of B not in A

▪ Notation: $A \subset B$ ▪ $A \subset B$ iff $A \subseteq B$ and

$A \neq B$ ▪ $A \subset B \rightarrow A \subseteq B$ as in example above

Example subsets

$A = \{1, 7, 9, 15\}$

$B = \{7, 9\}$

C = {7, 9, 15, 20}

True

B ⊆ C B ⊆ A B ⊂ A A ⊈ C 15 ∊ C {7, 9} ⊆ B {7} ⊂ A ∅ ⊆ C False

C ⊆ B {7, 9} ⊂ B A ⊆ B ∅⊂∅

A = { x | x ∊ ℕ and x ≥ 5} = {5, 6, 7, 8, …} B = {10, 12, 16, 20} C = { x | (∃ y)(y ∊ ℕ and x = 2y)} = {0, 2, 4, 6, …}

a. B ⊆ C true c. A ⊆ C false e. {11, 12, 13} ⊆ A true g. {12} ∊ B false (but 12 ∊ B) i. { x | x ∊ ℕ and x < 20} ⊈ B true k. { ∅} ⊆ B false (but ∅ ⊆ B)

Relationship of number sets

• ℕ ⊂ ℤ ⊂ ℚ ⊂ ℝ ⊂ ℂ

• Each number set is a proper subset of the next

• e.g., 86 ∈ ℕ, +86 ∈ ℤ, 86/1 ∈ ℚ, 86.0 ∈ ℝ, 86 + 0 i ∈ ℂ

• e.g., –86 ∈ ℤ but ∉ ℕ, 86/5 ∈ ℚ but ∉ ℤ, 86 π ∈ ℝ but ∉ ℚ, 86 + 12 i ∈ ℂ but ∉ ℝ

• Mnemonic: "Nine Zulu Queens Ruled China."

Proving set relationships

• May want to prove

▪ Subset A ⊆ B

▪ Proper subset A ⊂ B ▪ Equality A = B

• Proving subsets

▪ If the elements of a set have some property, the elements of a subset of the set have the property

▪ If B = { x | P (x)} and A ⊆ B, then (∀x)(x ∊ A → P (x))

▪ This can be used to prove A ⊆ B, A ⊂ B

• Proving equality ▪ To prove A = B, prove A ⊆ B and

B ⊆ A

Example subset proof

$B = \{ x \mid x$ is a multiple of $4\}$ $A = \{ x \mid x$ is a multiple of $8\}$ Theorem

$A \subseteq B$

Proof

Let $x \in A$. We must show $x \in B$. Because $x \in A$, $x = m \cdot 8$ for integer m. $x = m \cdot 8 = m \cdot 2 \cdot 4 = k \cdot 4$, where $k = 2$ m an integer. Therefore x is a multiple of 4, thus $x \in B$. ■ To prove $A \subset B$, prove $A \subseteq B$ and exhibit $x \in B$, $x \notin A$; e.g., $12 \in B$, $12 \notin A$.

Example subset proof $A = \{ x \mid x \in \mathbb{R}$ and $x 2 - 4 x + 3 = 0\} = \{1, 3\}$ $B = \{ x \mid x \in \mathbb{N}$ and $1 \leq x \leq 4\} = \{1, 2, 3, 4\}$

Theorem $A \subset B$

Proof

Let $x \in A$. We must show $x \in B$. Because $x \in A$, $x \in \mathbb{R}$ and $x 2 - 4x + 3 = 0$, so $x = 1$ or $x = 3$. In both cases, $x \in \mathbb{N}$ and

$1 \leq x \leq 4$, so $x \in B$, therefore $A \subseteq B$. $4 \in B$ but $4 \notin A$. Thus $A \subset B$. ■

Example set equality proof

$A = \{ x \mid x \in \mathbb{N}$ and $x 2 < 15\} = \{0, 1, 2, 3\}$ $B = \{ x \mid x \in \mathbb{N}$ and $2 x < 7\} = \{0, 1, 2, 3\}$

Theorem $A = B$

Proof

($A \subseteq B$) Let $x \in A$. Because $x \in A$, $x \in \mathbb{N}$ and $x 2 < 15$, so $x = 0, 1, 2,$ or 3 ($4 2 = 16 > 15$).

For each case, $2 x < 7$, so $x \in B$, therefore $A \subseteq B$. ($B \subseteq A$) Let $x \in B$.

Because $x \in B$, $x \in \mathbb{N}$ and $2 x < 7$, so $x = 0, 1, 2,$ or 3 ($2 \cdot 4 = 8 > 7$). For each case, $x 2 < 15$, so $x \in A$, therefore $B \subseteq A$.

$A \subseteq B$ and $B \subseteq A$, thus $A = B$.

Power sets

- Definition

▪ Set of all subsets (⊆) of a set

▪ Notation ℘ (A) ▪ e.g., A = {0, 1}, ℘ (A) = { ∅, {0}, {1}, {0, 1}} ▪ ℘ (A) is a set, elements of ℘ (A) are sets ▪ x ∊ ℘ (A) ↔ x ⊆ A • Size of ℘ (A) ▪ If A has n elements, ℘ (A) has 2 n elements ▪ Each element of A may be in or out of subset of A

Binary operations

• Samples

▪ Binary operation; addition x + y, subtraction 3 – 5 ▪ Unary operation: negation –8

• Concepts

▪ Operations are defined on sets (above on ℤ)

▪ Binary operation operands are ordered pairs, e.g., (3, 5)

• Definition

▪ Binary operation; ◦ is a binary operation on set S if for every ordered pair (x, y) of elements of S, x ◦ y exists, is unique, and is an element of S ▪ Well-defined; result exists and is unique

▪ Closed; result is in set S

Example valid binary operations Operation Symbol Set Exists Unique Closed addition + ℤ yes yes yes subtraction – ℤ yes yes yes multiplication · ℤ yes yes yes conjunction ∧ wffs yes yes yes disjunction ∨ wffs yes yes yes implication → wffs yes yes yes equivalence ↔ wffs yes yes yes

Example invalid binary operations

• Candidate ◦ for operation fails if for some x, y ∊ S ▪ x ◦ y does not exist (not defined) ▪ x ◦ y not unique ▪ x ◦ y not in S Operation Symbol Set Exists Unique Closed subtraction – ℕ yes yes no no name ◦ ℕ yes no yes division ÷ ℤ no yes no division ÷ ℤ – {0} yes yes no 1 – 2 = -1 ∉ ℕ 1 ÷ 0 not defined 1 ÷ 2 ∉ ℤ or ℤ – {0} 5 0 and 1 5if0 5if1 =—→= ⎩ ⎨ ⎧ ≤ ≥ = yxx x x yx

§4. Unique arrangements, permutations, combinations and their properties

What's the Difference?

In English we use the word "combination" loosely, without thinking if the **order** of things is important. In other words:

"My fruit salad is a combination of apples, grapes and bananas" We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.

"The combination to the safe is 472". Now we **do** care about the order. "724" won't work, nor will "247". It has to be exactly **4-7-2**.

So, in Mathematics we use more precise language:

- When the order doesn't matter, it is a **Combination**.
- When the order **does** matter it is a **Permutation**.

So, we should really call this a "Permutation Lock"!

In other words:

A Permutation is an **ordered** Combination.

To help you to remember, think "**P**ermutation ... **P**osition"

Permutations

There are basically two types of permutation:

- **Repetition is Allowed**: such as the lock above. It could be "333".
- **No Repetition**: for example the first three people in a running race. You can't be first and second.

1. Permutations with Repetition

These are the easiest to calculate.

When a thing has **n** different types ... we have **n** choices each time!

For example: choosing **3** of those things, the permutations are:

n × **n** × **n** (n multiplied 3 times)

More generally: choosing **r** of something that has **n** different types, the permutations are:

n × **n** × **... (r times)**

(In other words, there are **n** possibilities for the first choice, THEN there are **n** possibilites for the second choice, and so on, multplying each time.)

Which is easier to write down using an exponent of **r**:

n × **n** × **... (r times) =** $\mathbf{n^r}$

Example: in the lock above, there are 10 numbers to choose from (0,1,2,3,4,5,6,7,8,9) and we choose 3 of them:

10 × 10 × ... (3 times) = $\mathbf{10^3}$ **= 1,000 permutations**

So, the formula is simply:

$\mathbf{n^r}$

where **n** is the number of things to choose from,
and we choose **r** of them,
repetition is allowed,
and order matters.

2. Permutations without Repetition

In this case, we have to **reduce** the number of available choices each time.

Example: what order could 16 pool balls be in?

After choosing, say, number "14" we can't choose it again.

So, our first choice has 16 possibilites, and our next choice has 15 possibilities, then 14, 13, 12, 11, ... etc. And the total permutations are:

16 × 15 × 14 × 13 × ... = 20,922,789,888,000

But maybe we don't want to choose them all, **just 3** of them, and that is then:

16 × 15 × 14 = 3,360

In other words, there are 3,360 different ways that 3 pool balls could be arranged out of 16 balls.

Without repetition our choices get reduced each time.

But how do we write that mathematically? Answer: we use the "factorial function"

!

The **factorial function** (symbol: **!**) just means to multiply a series of descending natural numbers. Examples:

- 4! = 4 × 3 × 2 × 1 = 24
- 7! = 7 × 6 × 5 × 4 × 3 × 2 × 1 = 5,040
- 1! = 1

Note: it is generally agreed that **0! = 1**. It may seem funny that multiplying no numbers together gets us 1, but it helps simplify a lot of equations.

So, when we want to select **all** of the billiard balls the permutations are:

16! = 20,922,789,888,000

But when we want to select just 3 we don't want to multiply after 14. How do we do that? There is a neat trick: we divide by **13!**

16 × 15 × 14 × 13 × 12 × ...13 × 12 × ... = 16 × 15 × 14

That was neat: the **13 × 12 × ... etc** gets "cancelled out", leaving only **16 × 15 × 14**.

The formula is written:

n!**(n − r)!**

where **n** is the number of things to choose from,
and we choose **r** of them,
no repetitions,
order matters.

Example Our "order of 3 out of 16 pool balls example" is:

16!**(16−3)!** = 16!**13!** = 20,922,789,888,000**6,227,020,800** = 3,360

(which is just the same as: **16 × 15 × 14 = 3,360**)

Example: How many ways can first and second place be awarded to 10 people?

10!**(10−2)!** = 10!**8!** = 3,628,800**40,320** = 90

(which is just the same as: **10 × 9 = 90**)

Notation

Instead of writing the whole formula, people use different notations such as these:

$P(n,r) = {}^{n}P_{r} = {}_{n}P_{r} = n!\mathbf{(n-r)!}$

Examples:

- $P(10,2) = 90$
- ${}^{10}P_{2} = 90$
- ${}_{10}P_{2} = 90$

Combinations

There are also two types of combinations (remember the order does **not** matter now):

- **Repetition is Allowed**: such as coins in your pocket (5,5,5,10,10)
- **No Repetition**: such as lottery numbers (2,14,15,27,30,33)

1. Combinations with Repetition

Actually, these are the hardest to explain, so we will come back to this later.

2. Combinations without Repetition

This is how lotteries work. The numbers are drawn one at a time, and if we have the lucky numbers (no matter what order) we win!

The easiest way to explain it is to:

- assume that the order does matter (ie permutations),
- then alter it so the order does **not** matter.

Going back to our pool ball example, let's say we just want to know which 3 pool balls are chosen, not the order.

We already know that 3 out of 16 gave us 3,360 permutations.

But many of those are the same to us now, because we don't care what order!

For example, let us say balls 1, 2 and 3 are chosen. These are the possibilites:

Order does matter	Order doesn't matter
1 2 3	
1 3 2	
2 1 3	1 2 3
2 3 1	
3 1 2	
3 2 1	

So, the permutations have 6 times as many possibilites.

In fact there is an easy way to work out how many ways "1 2 3" could be placed in order, and we have already talked about it. The answer is:

3! = 3 × **2** × **1 = 6**

(Another example: 4 things can be placed in **4! = 4** × **3** × **2** × **1 = 24** different ways, try it for yourself!)

So we adjust our permutations formula to **reduce it** by how many ways the objects could be in order (because we aren't interested in their order any more):

n!**(n−r)!** x 1**r!** = n!**r!(n−r)!**

That formula is so important it is often just written in big parentheses like this:

n!**r!(n−r)!** = (n**r**)

where **n** is the number of things to choose from,
and we choose **r** of them,
no repetition,
order doesn't matter.

It is often called "n choose r" (such as "16 choose 3")

And is also known as the Binomial Coefficient.

Notation

All these notations mean "n choose r":

$C(n,r) = {}^{n}C_{r} = {}_{n}C_{r}$ = (n**r**) = n!**r!(n−r)!**

Just remember the formula:

n!**r!(n − r)!**

Example: Pool Balls (without order)

So, our pool ball example (now without order) is:

16!**3!(16−3)!**

= 16!**3! × 13!**

= 20,922,789,888,000**6 × 6,227,020,800**

= 560

Notice the formula 16!**3! × 13!** gives the same answer as 16!**13! × 3!**

So choosing 3 balls out of 16, or choosing 13 balls out of 16, have the same number of combinations:

16!**3!(16−3)!** = 16!**13!(16−13)!** = 16!**3! × 13!** = 560

In fact the formula is nice and **symmetrical**:

n!**r!(n−r)!** = (n**r**) = (n**n−r**)

Also, knowing that 16!/13! reduces to 16×15×14, we can save lots of calculation by doing it this way:

16×15×14**3×2×1**

= 3360**6**

= 560

Pascal's Triangle

We can also use Pascal's Triangle to find the values. Go down to row "n" (the top row is 0), and then along "r" places and the value there is our answer. Here is an extract showing row 16:

1 14 91 364 ...
1 15 105 455 1365 ...
1 16 120 **560** 1820 4368 ...

1. Combinations with Repetition

OK, now we can tackle this one ...

Let us say there are five flavors of icecream: **banana, chocolate, lemon, strawberry and vanilla**.

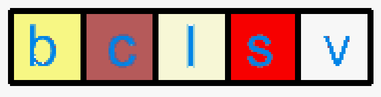
We can have three scoops. How many variations will there be?

Let's use letters for the flavors: {b, c, l, s, v}. Example selections include

- {c, c, c} (3 scoops of chocolate)
- {b, l, v} (one each of banana, lemon and vanilla)
- {b, v, v} (one of banana, two of vanilla)

(And just to be clear: There are **n=5** things to choose from, we choose **r=3** of them, order does not matter, and we **can** repeat!)

Now, I can't describe directly to you how to calculate this, but I can show you a **special technique** that lets you work it out.



Think about the ice cream being in boxes, we could say "move past the first box, then take 3 scoops, then move along 3 more boxes to the end" and we will have 3 scoops of chocolate!

So it is like we are ordering a robot to get our ice cream, but it doesn't change anything, we still get what we want.

We can write this down as →○○○→→→ (arrow means **move**, circle means **scoop**).

In fact the three examples above can be written like this:

{c, c, c} (3 scoops of chocolate):	→○○○→→→
{b, l, v} (one each of banana, lemon and vanilla):	○→→○→→○
{b, v, v} (one of banana, two of vanilla):	○→→→→○○

So instead of worrying about different flavors, we have a **simpler** question: "how many different ways can we arrange arrows and circles?"

Notice that there are always 3 circles (3 scoops of ice cream) and 4 arrows (we need to move 4 times to go from the 1st to 5th container).

So (being general here) there are **r + (n−1)** positions, and we want to choose **r** of them to have circles.

This is like saying "we have **r + (n−1)** pool balls and want to choose **r** of them". In other words it is now like the pool balls question, but with slightly changed numbers. And we can write it like this:

$$(r + n - 1)!\mathbf{r!(n - 1)!} = (r + n - 1\mathbf{r})$$

where **n** is the number of things to choose from, and we choose **r** of them repetition allowed, order doesn't matter.

Interestingly, we can look at the arrows instead of the circles, and say "we have **r + (n−1)** positions and want to choose **(n−1)** of them to have arrows", and the answer is the same:

(r + n − 1)!**r!(n − 1)!** = (r + n − 1**r**) = (r + n − 1**n − 1**)

So, what about our example, what is the answer?

(3+5−1)!**3!(5−1)!** = 7!**3!4!** = 5040**6×24** = 35

There are 35 ways of having 3 scoops from five flavors of icecream.

In Conclusion

Phew, that was a lot to absorb, so maybe you could read it again to be sure!

But knowing how these formulas work is only half the battle. Figuring out how to interpret a real world situation can be quite hard.

But at least you now know the 4 variations of "Order does/does not matter" and "Repeats are/are not allowed":

	Repeats allowed	**No Repeats**
Permutations (order matters):	n^r	n!**(n − r)!**
Combinations (order doesn't matter):	(r + n − 1)!**r!(n − 1)!**	n!**r!(n − r)!**

§5. Natural numbers. Definition of arithmetic operations on natural numbers and their properties based on "set theory"

Definition as von Neumann ordinals

In Zermelo–Fraenkel (ZF) set theory, the natural numbers are defined recursively by letting 0 = {} be the empty set and n + 1 = n ∪ {n} for each n. In this way n = {0, 1, …, n − 1} for each natural number n. This definition has the property that n is a set with n elements. The first few numbers defined this way are: (Goldrei 1996)

The set N of natural numbers is defined in this system as the smallest set containing 0 and closed under the successor function S defined by S(n) = n ∪ {n}. The structure ⟨N, 0, S⟩ is a model of the Peano axioms (Goldrei 1996). The existence of the set N is equivalent to the axiom of infinity in ZF set theory.

The set N and its elements, when constructed this way, are an initial part of the von Neumann ordinals.

Frege and Russell

Gottlob Frege and Bertrand Russell each proposed defining a natural number n as the collection of all sets with n elements. More formally, a natural number is an equivalence class of finite sets under the equivalence relation of equinumerosity. This definition may appear circular, but it is not, because equinumerosity can be defined in alternate ways, for instance by saying that two sets are equinumerous if they can be put into one-to-one correspondence—this is sometimes known as Hume's principle.

This definition works in type theory, and in set theories that grew out of type theory, such as New Foundations and related systems. However, it does not work in the axiomatic set theory ZFC nor in certain related systems, because in such systems the equivalence classes under equinumerosity are proper classes rather than sets.

Hatcher

William S. Hatcher (1982) derives Peano's axioms from several foundational systems, including ZFC and category theory, and from the system of Frege's Grundgesetze der Arithmetik using modern notation and natural deduction. The Russell paradox proved this system inconsistent, but George Boolos (1998) and David J. Anderson and Edward Zalta (2004) show how to repair it.

References

- Anderson, D. J., and Edward Zalta, 2004, "Frege, Boolos, and Logical Objects," Journal of Philosophical Logic 33: 1–26.
- George Boolos, 1998. Logic, Logic, and Logic.

Natural numbers

Natural numbers are a part of the number system which includes all the positive integers from 1 till infinity and are also used for counting purpose. It does not include zero (0). In fact, 1,2,3,4,5,6,7,8,9…., are also called counting numbers.

Natural numbers are part of real numbers, that include only the positive integers i.e. 1, 2, 3, 4,5,6, ………. excluding zero, fractions, decimals and negative numbers.

Note: Natural numbers do not include negative numbers or zero.

In this article, you will learn more about natural numbers with respect to their definition, comparison with whole numbers, representation in the number line, properties, etc.

Natural Number Definition

As explained in the introduction part, natural numbers are the numbers which are positive integers and includes numbers from 1 till infinity(∞). These numbers are countable and are generally used for calculation purpose. The set of natural numbers is represented by the letter "**N**".

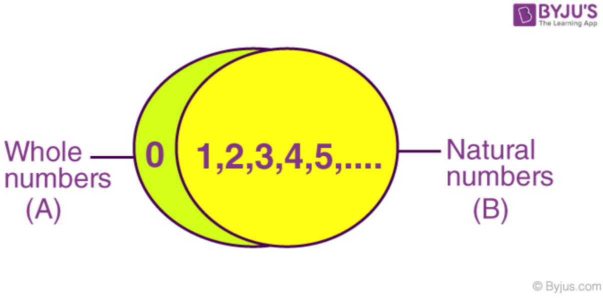
N = {1,2,3,4,5,6,7,8,9,10…….}

Natural Numbers and Whole Numbers

Natural numbers include all the whole numbers excluding the number 0. In other words, all natural numbers are whole numbers, but all whole numbers are not natural numbers.

- Natural Numbers = {1,2,3,4,5,6,7,8,9,…..}
- Whole Numbers = {0,1,2,3,4,5,7,8,9,….}

Check out the difference between natural and whole numbers to know more about the differentiating properties of these two sets of numbers.



The above representation of sets shows two regions. A ∩ B i.e. intersection of natural numbers and whole numbers (1, 2, 3, 4, 5, 6, ……..) and the green region showing A-B, i.e. part of the whole number (0).

Thus, a whole number is **"a part of Integers consisting of all the natural number including 0."**

Is '0' a Natural Number?

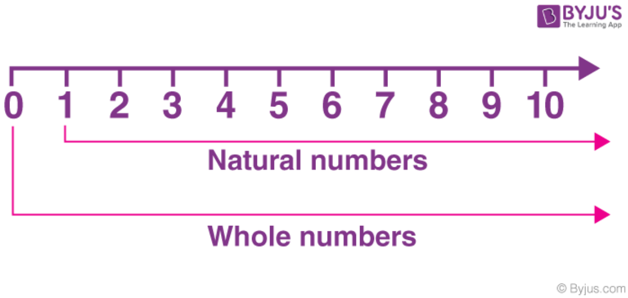
The answer to this question is 'No'. As we know already, natural numbers start with 1 to infinity and are positive integers. But when we combine 0 with a positive integer such as 10, 20, etc. it becomes a natural number. In fact, 0 is a whole number which has a null value.

Every Natural Number is a Whole Number. True or False?

Every natural number is a whole number. The statement is true because natural numbers are the positive integers that start from 1 and goes till infinity whereas whole numbers also include all the positive integers along with 0.

Representing Natural Numbers on a Number Line

Natural numbers representation on a number line is as follows:



The above number line represents natural numbers and whole numbers. All the integers on the right-hand side of 0 represent the natural numbers, thus forming an infinite set of numbers. When 0 is included, these numbers become whole numbers which are also an infinite set of numbers.

Set of Natural Numbers

In a set notation, the symbol of natural number is “N” and it is represented as given below.

Statement:

N = Set of all numbers starting from 1.

In Roster Form:

N = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ………………………………}

In Set Builder Form:

N = {x : x is an integer starting from 1}

Natural Numbers Examples

The natural numbers include the positive integers (also known as non-negative integers) and a few examples include 1, 2, 3, 4, 5, 6, …∞. In other words, natural numbers are a set of all the whole numbers excluding 0.

23, 56, 78, 999, 100202, etc. are all examples of natural numbers.

Properties of Natural Numbers

Natural numbers properties are segregated into four main properties which include:

- **Closure property**
- **Commutative property**
- **Associative property**
- **Distributive property**

Each of these properties is explained below in detail.

Closure Property

Natural numbers are always closed under addition and multiplication. The addition and multiplication of two or more natural numbers will always yield a natural number. In the case of **subtraction and division, natural numbers do not obey**

closure property, which means subtracting or dividing two natural numbers might not give a natural number as a result.

- **Addition:** 1 + 2 = 3, 3 + 4 = 7, etc. In each of these cases, the resulting number is always a natural number.
- **Multiplication:** 2 × 3 = 6, 5 × 4 = 20, etc. In this case also, the resultant is always a natural number.
- **Subtraction:** 9 – 5 = 4, 3 – 5 = -2, etc. In this case, the result may or may not be a natural number.
- **Division:** 10 ÷ 5 = 2, 10 ÷ 3 = 3.33, etc. In this case, also, the resultant number may or may not be a natural number.

Note: Closure property does not hold, if any of the numbers in case of multiplication and division, is not a natural number. But for addition and subtraction, if the result is a positive number, then only closure property exists.

For example:

- -2 x 3 = -6; Not a natural number
- 6/-2 = -3; Not a natural number

Associative Property

The **associative property holds true in case of addition and multiplication of natural numbers** i.e. a + (b + c) = (a + b) + c and a × (b × c) = (a × b) × c. On the other hand, for **subtraction and division of natural numbers, the associative property does not hold true**. An example of this is given below.

- **Addition:** a + (b + c) = (a + b) + c => 3 + (15 + 1) = 19 and (3 + 15) + 1 = 19.
- **Multiplication:** a × (b × c) = (a × b) × c => 3 × (15 × 1) = 45 and (3 × 15) × 1 = 45.
- **Subtraction:** a – (b – c) ≠ (a – b) – c => 2 – (15 – 1) = – 12 and (2 – 15) – 1 = – 14.

- **Division:** a ÷ (b ÷ c) ≠ (a ÷ b) ÷ c => 2 ÷(3 ÷ 6) = 4 and (2 ÷ 3) ÷ 6 = 0.11.

Commutative Property

For commutative property

- Addition and multiplication of natural numbers show the commutative property. For example, x + y = y + x and a × b = b × a
- Subtraction and division of natural numbers do not show the commutative property. For example, x – y ≠ y – x and x ÷ y ≠ y ÷ x

Distributive Property

- Multiplication of natural numbers is always distributive over addition. For example, a × (b + c) = ab + ac
- Multiplication of natural numbers is also distributive over subtraction. For example, a × (b – c) = ab – ac

Operations With Natural Numbers

An overview of algebraic operation with natural numbers i.e. addition, subtraction, multiplication and division, along with their respective properties are summarized in the table given below.

Properties and Operations on Natural Numbers

Operation	**Closure Property**	**Commutative Property**	**Associative Property**
Addition	Yes	Yes	Yes
Subtraction	No	No	No

Properties and Operations on Natural Numbers			
Multiplication	Yes	Yes	Yes
Division	No	No	No

Solved Examples

Question 1: Sort out the natural numbers from the following list: 20, 1555, 63.99, 5/2, 60, −78, 0, −2, −3/2

Solution: Natural numbers from the above list are 20, 1555 and 60.

Question 2: What are the first 10 natural numbers?

Solution: The first 10 natural numbers on the number line are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Question 3: Is the number 0 a natural number?

Solution: 0 is not a natural number. It is a whole number. Natural numbers only include positive integers.

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Frequently Asked Questions on Natural Numbers

What are Natural Numbers?

Natural numbers are the positive integers or non-negative integers which start from 1 and ends at infinity, such as:

1,2,3,4,5,6,7,8,9,10,……,∞.

Is 0 a Natural Number?

Zero does not have a positive or negative value. Since all the natural numbers are positive integers, hence we cannot say zero is a natural number. Although zero is called a whole number.

What are the first ten Natural Numbers?

The first ten natural numbers are: 1,2,3,4,5,6,7,8,9, and 10.

What is the difference between Natural numbers and Whole numbers?

Natural numbers include only positive integers and starts from 1 till infinity. Whereas whole numbers are the combination of zero and natural numbers, as it starts from 0 and ends at infinite value.

What are the examples of Natural numbers?

The examples of natural numbers are 5, 7, 21, 24, 99, 101, etc.

Definition of arithmetic operations on natural numbers and their properties based on "set theory".

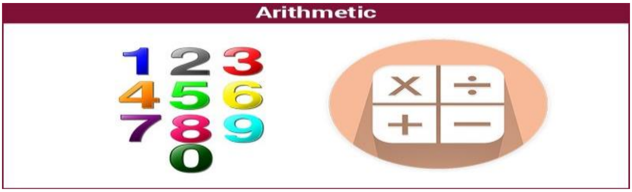
Arithmetic is one of the oldest and elementary branches of mathematics, originating from the Greek word ‘Arithmos’, which means ‘number’. Arithmetic involves the study of numbers, especially the properties of traditional operations, such as:

- Addition
- Subtraction
- Multiplication
- Division

The arithmetic operators based on these operations are ‘+’, ‘-’, ‘×’ and ‘÷’. Let us learn here all the important topics of arithmetic with examples.

What is Arithmetic?

Arithmetic is the fundamental of mathematics that includes the operations of numbers. These operations are addition, subtraction, multiplication and division. Arithmetic is one of the important branches of mathematics, that lays the foundation of the subject ‘Maths’, for students.



History of Arithmetic

The Fundamental principle of number theory was provided by Carl Friedrich Gauss in 1801, according to which, any integer which is greater than 1 can be described as the product of prime numbers in only one way. Arithmetic is another name given to number theory. The four elementary operations in arithmetic are addition, subtraction, multiplication and division. All these operations are discussed in brief here.

Arithmetic operations

The basic operations under arithmetic are addition and subtraction, division and multiplication, although the subject involves many other modified operations.

Addition (+)

Addition is among the basic operations in arithmetic. In simple forms, addition combines two or more values into a single term, for example: 2 + 5 = 7, 6 + 2 = 8, where ‘+’ is the addition operator.

The procedure of adding more than two values is called summation and involves methods to add n number of values.

The identity element of addition is 0, which means that adding 0 to any value gives the same result. The inverse element of addition is the opposite of any value, which means that adding the opposite of any digit to the digit itself gives the additive identity. For instance, the opposite of 5 is -5, therefore 5 + (-5) = 0.

Examples of addition:

- 8 + 10 = 18
- 12 + 5 = 17

Subtraction (−)

Subtraction can be labelled as the inverse of addition. It computes the difference between two values, i.e., the minuend minus the subtrahend. The operator of subtraction is (-).

If the minuend is greater than the subtrahend, the difference is positive. If the minuend is less than the subtrahend, the result is negative, and 0 if the numbers are equal.

For example,

- 4 – 3 = 1
- 3 – 4 = -1

Multiplication (×)

Multiplication also combines two values like addition and subtraction, into a single value or product. The two original values are known as the multiplicand and the multiplier, or simply both as factors.

The product of a and b is expressed as a·b or a × b, where ‘×’ is the multiplication operator. In software languages wherein only characters are used that are found in keyboards, it is often expressed as, a*b (* is called asterisk).

For example,

- 4 × 5 = 20
- 2 × 3 = 6

Division (÷)

The division is the inverse of multiplication. The operator used for the division method is '÷' or '/' sometimes. It computes the quotient of two numbers, the dividend that is divided by the divisor.

The quotient is more than 1 if the dividend is greater than the divisor for any well-defined positive number else, it is smaller than 1.

For example,

- $10 \div 2 = 5$
- $9 \div 3 = 3$

What is Arithmetic Sequence?

An arithmetic sequence is a sequence of numbers, where the difference between one term and the next is a constant. For example, 1, 4, 7, 10, 13, 16, 19, 22, 25, … is an arithmetic sequence with common difference equal to 3. It is also termed arithmetic progression and is commonly represented as:

$a, a + d, a + 2d, a + 3d, a + 4d, \ldots\ldots\ldots, a + (n – 1) d$

Where,

a = first term

d = common difference between the terms

n = number of terms

Arithmetic Solved Problems

Question 1: The sum of the two numbers is 50, and their difference is 30. Find the numbers.

Solution: Let the numbers be x and y. Now, as per the given situation,

$x + y = 50$……………………(i)

and $x – y = 30$………………(ii)

We can write, $x = 50-y$, from eq.(i),

Therefore, putting the value of x in eq(ii), we get,

$50 – y – y = 30$

$50 -2y = 30$

$2y = 50-30= 20$

$y = 20/2 = 10$

and $x = 50 – y = 50-10 =40$

Therefore, the two numbers are 40 and 10.

Question 2: Solve 25 + 5 (27 ÷ 3) – 9.

Solution: $25 + 5(27 \div 3) – 9$

$\Rightarrow 25 + 5(9) – 9$

$\Rightarrow 25 + 45 – 9$

$\Rightarrow 70 – 9 = 61$

Practice Questions on Arithmetic

1. Solve: 4 + 3 x 10 – 1
2. Solve: 246 x 132
3. Solve: 9/10 – 3/20
4. 65 is what percent of 500?

Frequently Asked Questions

What is Arithmetic in Maths?

Arithmetic is one of the branches of mathematics which deals with different types of numbers like odd numbers, whole numbers, even numbers, etc. and their basic operations involve addition, subtraction, multiplication, and division.

What are the arithmetic operators?

The four basic arithmetic operators are addition (+), subtraction (-), multiplication (×) and division (÷).

What are the Properties of Operation in Arithmetic?

There are four main properties of operations which include:

- Commutative Property
- Associative Property

- Distributive Property
- Additive Identity

What is the Order of Operations?

The BODMAS or PEMDAS rule is followed to order any operation involving +, −, ×, and ÷. The order of operation is:

B: Brackets
O: Order
D: Division
M: Multiplication
A: Addition
S: Subtraction

§6. The emergence and development of number systems. Positional and positional number systems. Decimal number system. Division relation on the set of non-negative integers. Divisors and divisors of natural numbers

A number is a way to represent arithmetic value, count or measure of a particular quantity. A **number system** can be considered as a mathematical notation of numbers using a set of digits or symbols. In simpler words the number system is a method of representing numbers. Every number system is identified with the help of its base or radix.

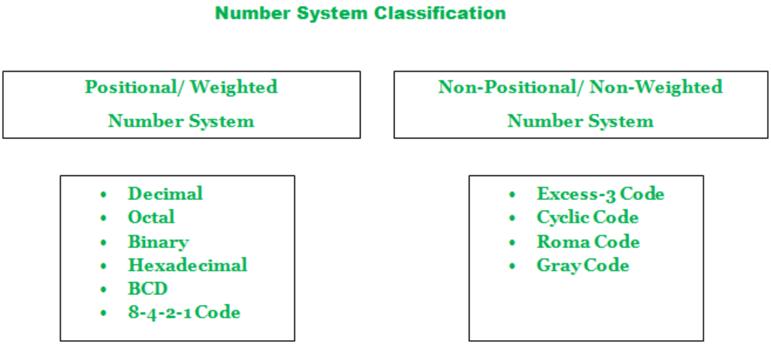
Base or Radix of a Number System: The base or radix of a number system can be referred as the total number of different symbols which can be used in a particular number system. Radix means “root” in Latin.

Base equals to 4 implies there are 4 different symbols in that number system. Similarly, base equals to “x” implies there are “x” different symbols in that number system.

Classification of Number System:

The number system can be classified in to two types namely:

Positional and Non-Positional number system



1. Positional (or Weighted) Number System:

A positional number system is also known as weighted number system. As the name implies there will be a weight associated with each digit.

According to its position of occurrence in the number, each digit is weighted. Towards the left the weights increases by a constant factor equivalent to the base or radix. With the help of the radix point ('.'), the positions corresponding to integral weights (1) are differentiated from the positions corresponding to fractional weights (<1).

Any integer value that is greater than or equal to two can be used as the base or radix. The digit position 'n' has weight

. Largest value of digit position is always 1 less than the base value. The value of a number is weighted sum of its digits.

For example:

1358 = 1 x + 3 x + 5 x + 8 x

13.58 = 1 x + 3 x + 5 x + 8 x

Few examples of positional number system are decimal number system, Binary number system, octal number system, hexadecimal number system, BCD, etc.

2. Non-Positional (or Non-weighted) Number System: Non-positional number system is also known as non-weighted number system. Digit value is independent of its position. Non-positional number system is used for shift position encodes and error detecting purpose.

Few examples of non-weighted number system are gray code, roman code, excess-3 code, etc.

Decimal number system

In Mathematics, the numbers can be classified into different types, namely real numbers, natural numbers, whole numbers, rational numbers, and so on. Decimal numbers are among them. It is the standard form of representing integer and non-integer numbers. In this article, let us discuss in detail about " **Decimals**", its types, properties, and place value representation of decimal numbers with many solved examples.

What are Decimals?

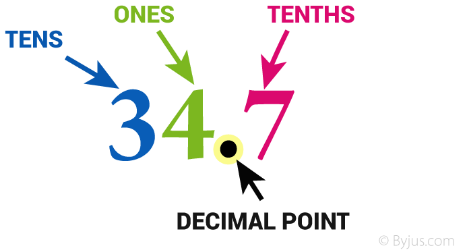
In Algebra, decimals are one of the types of numbers, which has a whole number and the fractional part separated by a decimal point. The dot present between the whole number and fractions part is called the decimal point. For example, 34.5 is a decimal number.

Here, 34 is a whole number part and 5 is the fractional part.

"." is the decimal point.

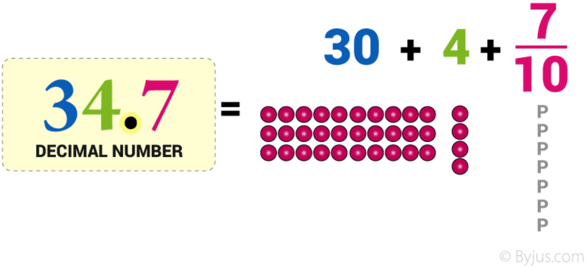
Let us discuss some other examples.

Here is the number "thirty-four and seven-tenths" written as a decimal number:



The decimal point goes between Ones and Tenths

34.7 has 3 Tens, 4 Ones and 7 Tenths



Types of Decimal Numbers

Decimal Numbers may be of different kinds, namely

Recurring Decimal Numbers (Repeating or Non-Terminating Decimals)

Example-

3.125125 (Finite)

3.121212121212….. (Infinite)

Non-Recurring Decimal Numbers (Non Repeating or Terminating Decimals):

Example:

3.2376 (Finite)

3.137654….(Infinite)

Decimal Fraction- It represents the fraction whose denominator in powers of ten.

Example:

81.75 = 8175/100

32.425 = 32425/1000

Converting the Decimal Number into Decimal Fraction:

For the decimal point place “1” in the denominator and remove the decimal point.

“1” is followed by a number of zeros equal to the number of digits following the decimal point.

For Example:

8 1 . 7 5

↓ ↓ ↓

1 0 0

81.75 = 8175/100

8 represents the power of 10^1 that is the tenths position.

1 represents the power of 10^0 that is the units position.

7 represents the power of 10^{-1} that is the one-tenths position.

5 represents the power of 10^{-2} that is the one-hundredths position.

So that is how each digit is represented by a particular power of 10 in the decimal number.

Place Value in Decimals

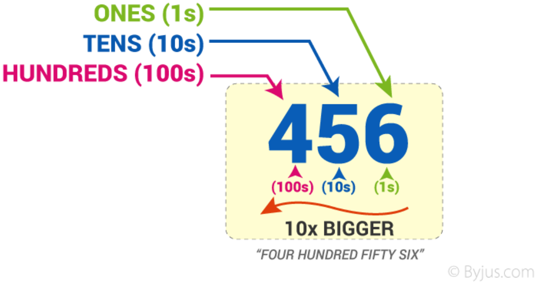
The place value system is used to define the position of a digit in a number which helps to determine its value. When we write specific numbers, the position of each digit is important.

Example:

For instance, let’s consider the number 456.

- The position of “6” is in One’s place, which means 6 ones (i.e. 6).
- The position of “5” is in the Ten’s place, which means 5 tens (i.e. fifty).
- The position of “4” is in the Hundred’s place, which means 4 hundred.
- As we go left, each position becomes ten times greater.

Hence, we read it as “Four hundred fifty-six”.



As we move left, each position is **10 times bigger!**

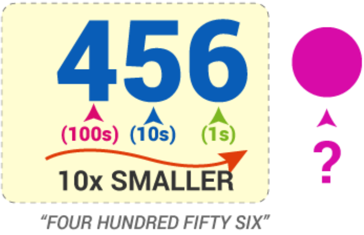
Tens are 10 times bigger than Ones.

Hundreds are 10 times bigger than Tens.

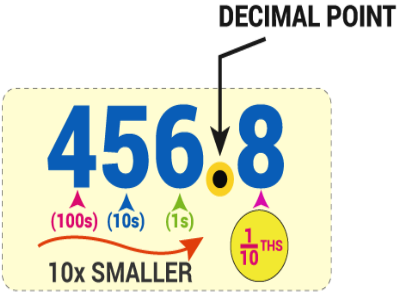
And

Each time we move right every position becomes **10 times smaller** from Hundred’s to Ten’s, to Ones

But if we continue past Ones?

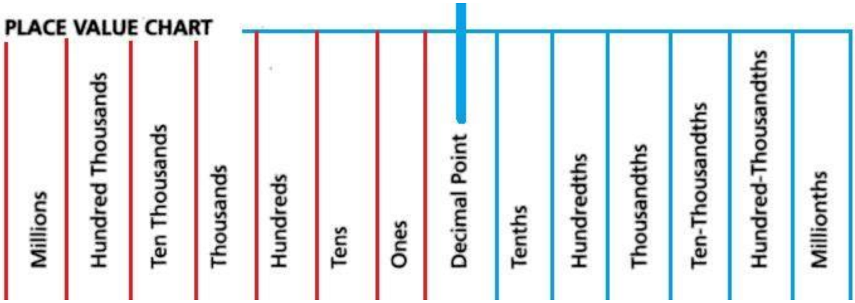


What are 10 times smaller than Ones?
(Tenths) are!
Before that, we should first put a decimal point,
So we already know that where we put that decimal point.



We say the above example as four hundred and fifty-six and eight-tenths but we usually just say four hundred and fifty-six point eight.

The power of 10 can be found using the following Place Value Chart:



The digits to the left of the decimal point are multiplied with the positive powers of ten in increasing order from right to left.

The digits to the right of the decimal point are multiplied with the negative powers of 10 in increasing order from left to right.

Following the same example 81.75

The decimal expansion of this is :

{(8*10)+(1*1)} + {(7*0.1)+(5*0.01)}

Where each number is multiplied by its associated power of ten.

Properties of Decimals

The important properties of decimal numbers under multiplication and division operations are as follows:

- If any two decimal numbers are multiplied in any order, the product remains the same.
- If a whole number and a decimal number are multiplied in any order, the product remains the same.
- If a decimal fraction is multiplied by 1, the product is the decimal fraction itself.
- If a decimal fraction is multiplied by 0, the product is zero (0).

- If a decimal number is divided by 1, the quotient is the decimal number.
- If a decimal number is divided by the same number, the quotient is 1.
- If 0 is divided by any decimal, the quotient is 0.
- The division of a decimal number by 0 is not possible, as the reciprocal of 0 does not exist.

Arithmetic Operation on Decimals

Like integers, the arithmetic operations like addition, subtraction, multiplication, and division operation can be performed on decimal numbers. Now, let us discuss important tips while performing arithmetic operations.

Addition

While adding decimal numbers, line up the decimal points of the given numbers and add the numbers. If a decimal point is not visible (i.e., whole numbers), the decimal is behind the number.

Subtraction

Similar to the addition of decimal numbers, line up the decimal point of the given numbers, and subtract the values. To perform the arithmetic operation, use place holding zeros for our reference.

Multiplication

Multiply the given numbers like integers, as if the decimal point is not present. Find the product and count up how many numbers are present after the decimal point in both the numbers. The count represents how many numbers are required after the decimal point in the product value.

Division

To simply divide the decimal numbers, move the decimal point in the numbers such that the number becomes the whole numbers. Now, perform the division operation like the integer division.

Decimal to Fraction Conversion

The conversion of decimal to fraction or fraction to decimal values can be performed easily. Now, we will discuss both the conversion methods below:

Decimal to Fraction Conversion

We know the numbers after the decimal point represents the tenths, hundredths, thousandths, and so on. Thus, while converting decimal to fraction, write down the decimal numbers in the expanded form and simplify the values

For example, 0.75

The expanded form of 0.75 is 75 x (1/100) = 75/100 = 3/4.

Fraction to Decimal Conversion

To convert the fraction to the decimal, simply divide the numerator by denominator.

For example, 7/2 is a fraction. If it's divided, we get 3.5.

Decimal Problems

Example 1:

Convert 8/10 in decimal form.

Solution:

To convert fraction to decimal, divide 8 by 10, we get the decimal form.

Thus, 8/10 = 0.8

Hence, the decimal form of 8/10 is 0.8

Example 2:

Express 1.25 in fraction form.

Solution:

The given decimal number is 1.25

The expanded form of 1.25 is
= 125 x (1/100)
= 125 /100
=5/4
Hence, the equivalent fraction for 1.25 is 5/4.

Frequently Asked Questions on Decimals

What is meant by decimals?

Decimals are the numbers, which consist of two parts namely, a whole number part and a fractional part separated by a decimal point. For example, 12.5 is a decimal number.

What are the different types of decimals?

The two different types of decimals are:
Terminating decimals (or) Non-recurring decimals
Non-terminating decimals (or) Recurring decimals

How to convert fractions to decimals?

To convert fractions to decimals, divide the numerator by the denominator value.

Write the expanded form of 74.2?

The expanded form of 74. 2 is 70 + 4 + (2/10).

When do we use decimals?

The decimal numbers are used when the problem requires more precision than the whole value. For example, while dealing with weight, money, length, and so on.

To learn more about decimals, division of decimals, and operations of converting fractions to decimals, Register with BYJU’S, and strengthen your skills.

Divisors and divisors of natural numbers

What are Divisors?

For any natural number N, the divisors (also known as factors) are those numbers that divide the number N without leaving a remainder. For example,

- Divisors of 15 are 1, 3, 5, 15
- All divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24

How to find the Divisors of a number?

The key here is the prime factors of that number.

For example, let’s write all the divisors of 24. 24 in terms of its prime factors is expressed as 23∗3123∗31. As written below, the number of divisors of 24 are 8:

1 = 2020 x 3030
2 = 2121 x 3030
3 = 2020 x3131
4 = 2222 x 3030
6 = 2121 x 3131
8 = 2323 x 3030
12 = 2222 x 3131
24 = 2323 x 3131

Now that you have brushed up your basic skills on divisors, let’s look into the formulae used to solve the problems on the divisors of a number along with examples.

Number of Divisors of a number

If N is a composite number such that

N=ap∗bq∗cr…N=ap∗bq∗cr…

where, a, b, and c are prime numbers, then the number of divisors of N is given by the formula:

(p+1)(q+1)(r+1)…(p+1)(q+1)(r+1)…

Example 1: Find the number of Divisors of 18

First lets express 18 in terms of prime factors:

18=21∗3218=21∗32

Here,

1. Consider the exponents of the prime factors of 18.

- The exponents of prime factors 2, and 3 are 1, and 2 respectively.

2. Increment each of the exponent obtained in previous step by 1.

3. Product of the incremented exponents from step 2 gives the solution to our problem. i.e (1+1)(2+1)=6(1+1)(2+1)=6

Hence the number of divisors of 18 is 6.

Example 1: Find the number of Divisors of 360

Example 2: Find the number of Divisors of 1728

Number of Even and Odd Divisors of a number

360=23∗32∗5360=23∗32∗5

In this case, except for the exponent of 2, we increment the exponent of all other factors by 1 and take their product along with the non-incremented exponent of 2. Thus we have,

Number of even factors of 360 => 3 x (2 + 1) x (1 + 1) = 18

To calculate the number of odd divisors of 360, ignore the exponent of 2 and take the product of the incremented(increment by 1) exponents of all other prime factors. Therefore,

Number of odd divisors of 360 => (2+1)(1+1) = 6

In general, if N is a composite number such that N=2p∗bq∗cr…N=2p∗bq∗cr…

where, a, b, and c are prime numbers, then

The Number of Even Divisors or the Even Factors of N is given by

(p)(q+1)(r+1)…(p)(q+1)(r+1)…

Example 1: Find the number of even factors of 3360

Express 3360 in terms of its prime factors:

$3360=25\ast31\ast51\ast713360=25\ast31\ast51\ast71$

As explained above, to calculate the even factors of 3360, except for the power of 2 we increment the exponent of all other factors by 1 and take their product along with the power of 2.

Thus, the number of even factors of 3360 => 5(1+1)(1+1)(1+1) => 40

Example 2: Find the number even divisors of 10800

Example 3: Find the number even factors of 12600

The Number of Odd Divisors or the Odd Factors of N is given by

$(q+1)(r+1)\ldots(q+1)(r+1)\ldots$

Example 1: Find the number of odd factors of 84

Write down 84 in terms of its prime factors:

$84=22\ast31\ast7184=22\ast31\ast71$

As explained above, to calculate the odd factors of 84, except for the power of 2 we increment the exponent of all other factors by 1 and take their product.

Thus, the number of odd factors of 84 => (1+1)(1+1)}=> 4

Example 2: Find the number odd divisors of 252

Example 3: Find the number odd divisors of 360

Example 4: Find the number odd divisors of 98000

Sum of Divisors of a number

For $N=ap\ast bq\ast cr\ldots N=ap\ast bq\ast cr\ldots$, Sum of the divisors of N is given by the formula

$(ap+1-1a-1)(bq+1-1b-1)(cr+1-1c-1)\ldots(ap+1-1a-1)(bq+1-1b-1)(cr+1-1c-1)\ldots$

Example 1: Find the sum of divisors of 18

First write down 18 in terms of its prime factors

21∗3221∗32

Based on the above formula, the Sum of divisors of 18=>

(21+1−12−1)(32+1−13−1)=39(21+1−12−1)(32+1−13−1)= 39

Example 2: Find the sum of divisors of 360

Example 3: Find the sum of divisors of 544

Product of Divisors of a number

For N=ap∗bq∗cr…N=ap∗bq∗cr…, the formula for product of divisors of N is given by

Nx2, where x = Number of divisors of N2Nx2, where x = Number of divisors of N2

Example 1: Find the product of factors of 360

For 360, we have previously found the number of factors of 360 to be 24. Now, the product of divisors of 360 is

360242=>36012360242=>36012

Example 2: Find the product of divisors of 512

Example 3: Find the product of divisors of 7056

Sum of odd divisors of a number

Now how do we find the sum of odd divisors of a number

This is basically similar to calculating the sum of divisors of a number. Let's look into this with the help of an example.

How to find the sum of odd divisors of 360?

First express 360 using its prime factors using the formula for sum of divisors of a number:

23∗32∗523∗32∗5

Now, the Sum of odd divisors of 360 is obtained by ignoring the powers of 2 => 32∗532∗5

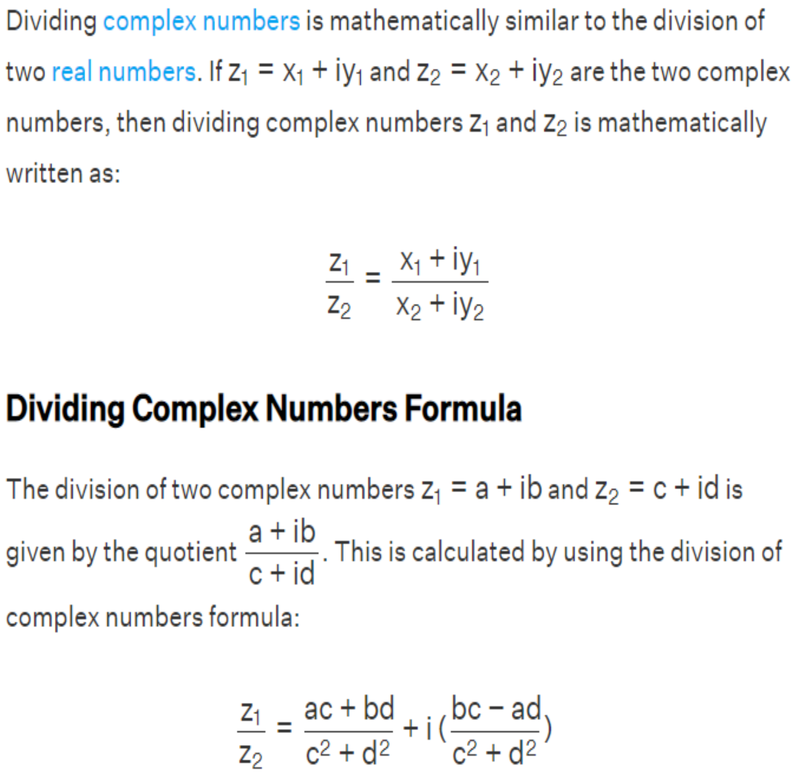
(20+1−12−1)(32+1−13−1)(51+1−15−1)=78

§7. Simple and complex numbers. Division signs in the decimal number system. Signs of division independent of the base of the number system: signs of division of sum, difference, product and quotient. Signs of division depending on the basis of the number system: 2, 5, 10, 3, 9, 4, 25, 100, 8, 125 in the decimal number system signs of division

Dividing complex numbers is a little more complicated than addition, subtraction, and multiplication of complex numbers because it is difficult to divide a number by an imaginary number. For **dividing complex numbers**, we need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary part of the denominator so that we end up with a real number in the denominator.

In this article, we will learn about the division of complex numbers, dividing complex numbers in polar form, the division of imaginary numbers, and dividing complex fractions.

What is Dividing Complex Numbers?



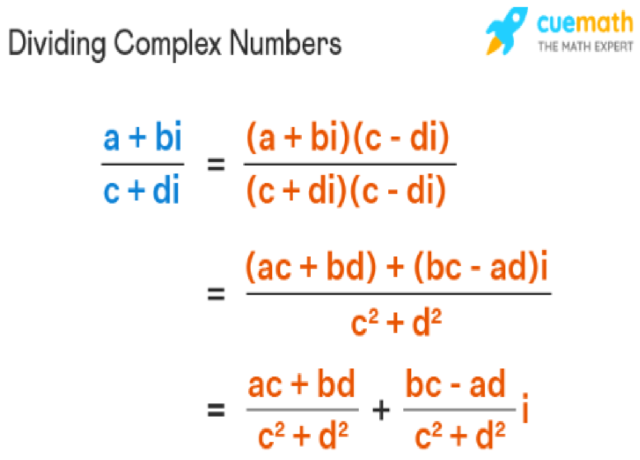
Dividing complex numbers is mathematically similar to the division of two real numbers. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are the two complex numbers, then dividing complex numbers z_1 and z_2 is mathematically written as:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$

Dividing Complex Numbers Formula

The division of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is given by the quotient $\frac{a + ib}{c + id}$. This is calculated by using the division of complex numbers formula:

$$\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + i\left(\frac{bc - ad}{c^2 + d^2}\right)$$



Dividing Complex Numbers

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)}$$

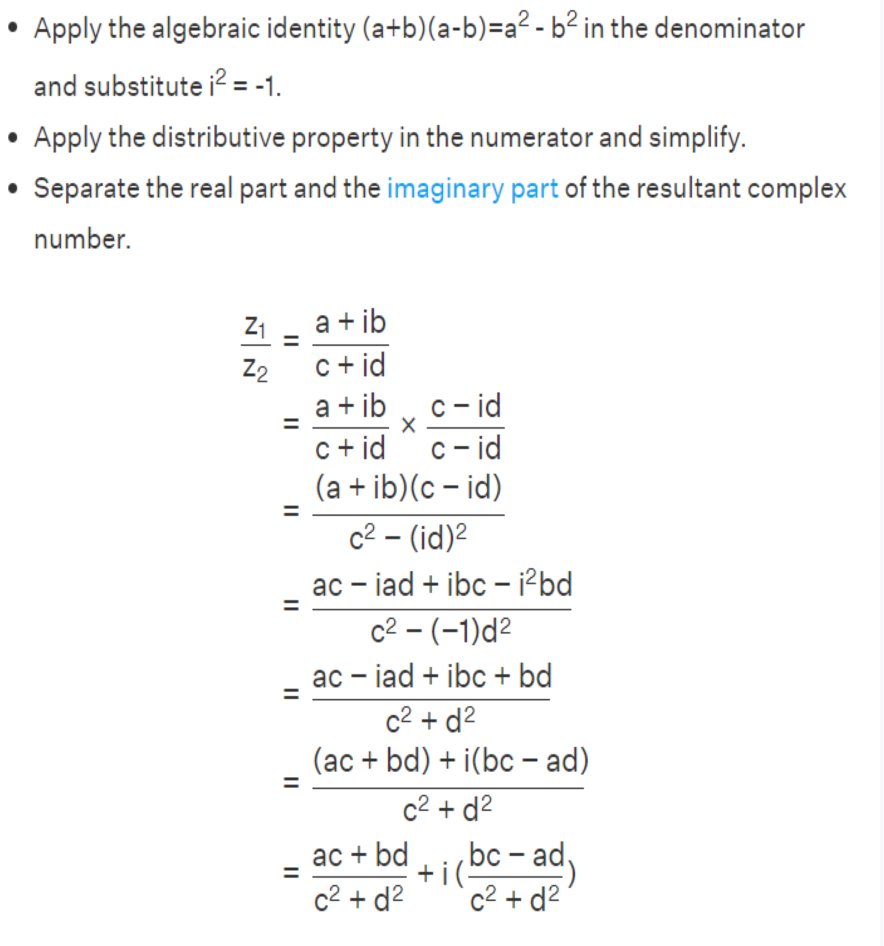
$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Steps for Dividing Complex Numbers

Now, we know what dividing complex numbers is, let us discuss the steps for dividing complex numbers. To divide the two complex numbers, follow the given steps:

- First, calculate the conjugate of the complex number that is at the denominator of the fraction.
- Multiply the conjugate with the numerator and the denominator of the complex fraction.



- Apply the algebraic identity (a+b)(a-b)=$a^2 - b^2$ in the denominator and substitute $i^2 = -1$.
- Apply the distributive property in the numerator and simplify.
- Separate the real part and the imaginary part of the resultant complex number.

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{a + ib}{c + id} \\
&= \frac{a + ib}{c + id} \times \frac{c - id}{c - id} \\
&= \frac{(a + ib)(c - id)}{c^2 - (id)^2} \\
&= \frac{ac - iad + ibc - i^2bd}{c^2 - (-1)d^2} \\
&= \frac{ac - iad + ibc + bd}{c^2 + d^2} \\
&= \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} \\
&= \frac{ac + bd}{c^2 + d^2} + i\left(\frac{bc - ad}{c^2 + d^2}\right)
\end{aligned}$$

How do we divide complex numbers?

Dividing a complex number by a real number is simple. For example:

$$\frac{2+3i}{4} = \frac{2}{4} + \frac{3}{4}i$$

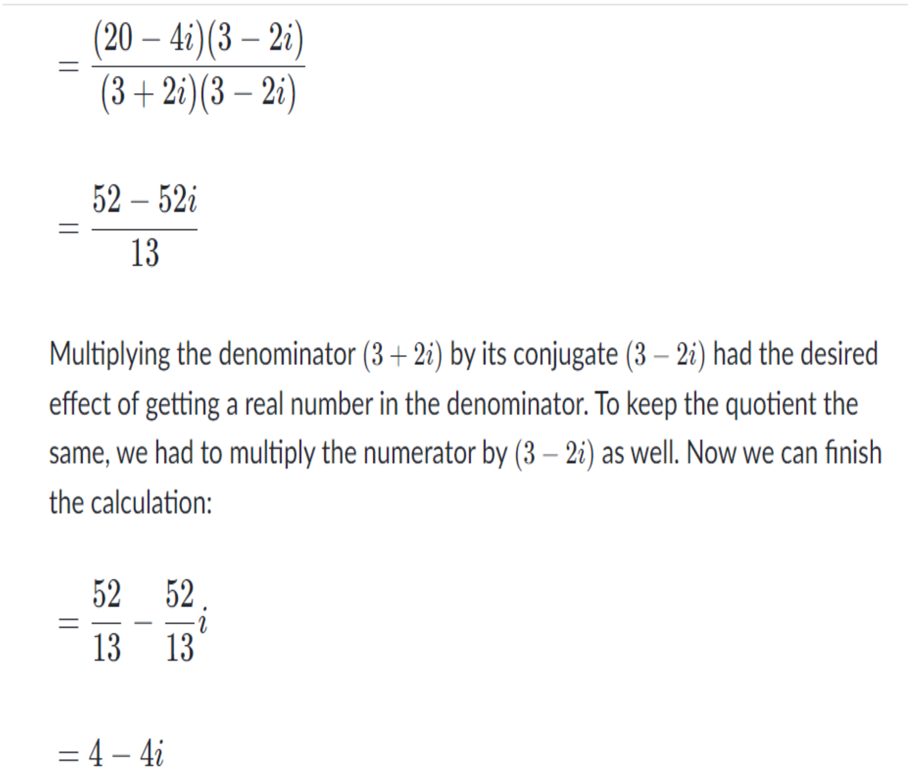
$$= 0.5 + 0.75i$$

Finding the quotient of two complex numbers is more complex (haha!). For example:

$$\frac{20-4i}{3+2i}$$

$$= \frac{20-4i}{3+2i} \cdot \frac{3-2i}{3-2i}$$

We multiplied both sides by the **conjugate** of the denominator, which is a number with the same real part and the opposite imaginary part. What's neat about conjugate numbers is that their product is *always* a real number. Let's continue:



$$= \frac{(20-4i)(3-2i)}{(3+2i)(3-2i)}$$

$$= \frac{52-52i}{13}$$

Multiplying the denominator $(3+2i)$ by its conjugate $(3-2i)$ had the desired effect of getting a real number in the denominator. To keep the quotient the same, we had to multiply the numerator by $(3-2i)$ as well. Now we can finish the calculation:

$$= \frac{52}{13} - \frac{52}{13}i$$

$$= 4 - 4i$$

Signs of division depending on the basis of the number system: 2, 5, 10, 3, 9, 4, 25, 100, 8, 125 in the decimal number system signs of division

DIVISIBILITY BY 2

A number that is divisible by 2 is called an even number. When the last digit in a number is 0 or even—that is, 2, 4, 6, or 8—then the number is divisible by 2. For example, 20 ends in a 0 so it’s divisible by 2. The number 936 ends in a 6, and 6 is even. So 936 is divisible by 2.

DIVISIBILITY BY 3

A number is divisible by 3 if the sum of the digits is divisible by 3. To use this trick, students must have some ability to divide, but checking smaller numbers is less daunting than a larger one. For instance, if you ask students if 168 is divisible by 3, they should do the following:

1 + 6 + 8 = 15

15/3 = 5

Therefore, 168 is divisible by 3.

DIVISIBILITY BY 4

If the last two digits of a number are divisible by 4, the whole number is. For example, in 1,012, 12 is divisible by 4. However, in 1,013, 13 is not. Therefore, 1,012 is divisible by 4 but 1,013 is not.

DIVISIBILITY BY 5

When the last digit of a number is 0 or 5, the number can evenly be divided by 5. As such, 5, 10, 15, 20, 25 and so on can all be divided by 5. Students can look at large numbers and say right away whether it can evenly be divided into five parts.

DIVISIBILITY BY 6

Numbers divisible by 6 can also be divided by both 3 and 2. Students should test the number with both rules for 3 and 2. If the number passes both tests, it can be divided by 6. If it fails just one test it cannot. For instance:

308 ends in an even digit, so it’s divisible by 2. However, 3 + 0 + 8 = 11, which cannot be divided evenly by 3. As such, 308 is not divisible by 6.

DIVISIBILITY BY 8

A large number is divisible by 8 if the last three digits are also divisible by 8 or are 000. In 7,120, 120 can be divided evenly by 8, so the 7,120 is divisible by 8 as well.

DIVISIBILITY BY 9

The divisibility rule for 9 is the same as for 3. If the sum of a number's digits is divisible by 9, so too is the entire number. For example:

In 549, 5 + 4 + 9 = 18

18/9 = 2

So, 549 is divisible by 9.

DIVISIBILITY BY 10

If the last digit is 0, the number can be divided evenly by 10.

Why the Rules Help and How to Use Them

These rules allow students to look at larger numbers in a less-daunting context. Divisibility rules also let them learn a lot about a number by simply looking at its digits. As such, you should encourage students to use all rules when examining a number. When looking at something like 1,159,350, students can go down the divisibility list, checking off which numbers the large one can be divided by.

Of course, you won't only talk about even division in your math class. Some numbers will have remainders. You can still use the rules to talk about those numbers. Have students determine whether a certain number will have a remainder when divided by 2, 3, 4, 5, 6, 8 or 10.

§8. Understanding fractions. Types of fractions with a unit value. Comparison of fractions, properties. Concept of mixed number. Converting a mixed number to an improper fraction and vice versa: converting an improper fraction to a mixed number

Before exploring the **types of fractions**, let us recall fractions. A fraction is a portion or part of any quantity out of a whole, where the whole can be any number, a specific value, or a thing. In many real-time situations, each and every quantity to be measured cannot be an absolute whole number. Hence, we may have to deal with parts of a whole or portions of a whole. This is where the concept of fractions comes in. In this lesson, let us learn about **different types of fractions** with examples, such as proper and improper fractions, mixed fractions, equivalent fractions, like and unlike fractions.

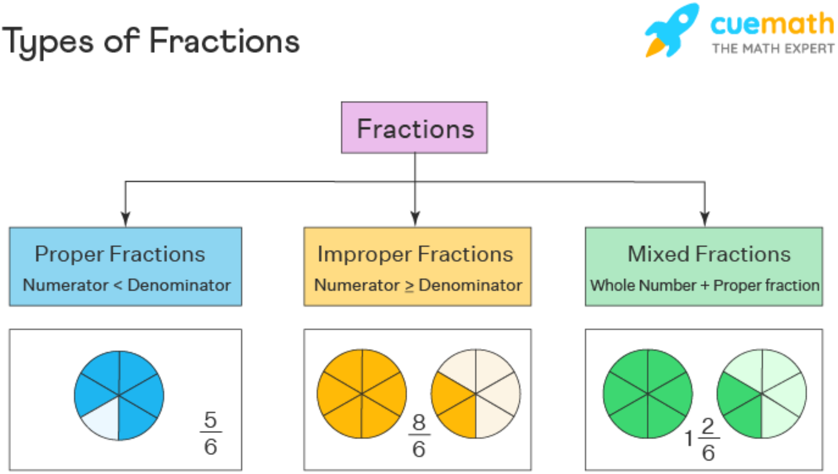
What are the Different Types of Fractions?

The **different types of fractions** are distinguished mostly on the basis of their numerator and denominator. A fraction consists of two parts, the numerator, and the denominator. The numerator is the number that is placed on the top of the fractional bar, while the number that is placed on the bottom is called the denominator. The numerator indicates the number of parts that are being considered, whereas, the denominator indicates the total number of parts in the whole.

Although there are many types of fractions, the three main types of fractions that are differentiated on the basis of the numerator and the denominator are:

- Proper fractions
- Improper fractions

- Mixed fractions



Proper Fractions

A fraction whose numerator is less than its denominator is called a proper fraction. For example, 3/12 and 2/5 are proper fractions because 3 < 12 and 2 < 5. Example: Sam got a bar of chocolate and he divided it into 3 equal parts. He took 1 part and gave 2 parts to his sister, Sara. We can represent Sam's portion as 1/3 and Sara's portion as 2/3. Both these fractions are considered proper fractions.

Improper Fractions

A fraction whose numerator is greater than or equal to its denominator is called an improper fraction. For example, 5/2 and 8/7 are improper fractions because 5 > 2 and 8 > 7.

Mixed Fractions

A mixed fraction is a mix of a whole number and a proper fraction. For example, 134134 and 347347 are mixed numbers or mixed fractions. In the first example, 1 is the whole number part and 3/4 is the proper fraction. In the second example, 3 is the whole number part and 4/7 is the proper fraction.

Now, let us study the types of fractions that are classified in groups. When a **group of fractions** is classified they help in comparing fractions. They are categorized as follows:

- **Like fractions**
- **Unlike fractions**
- **Equivalent fractions**

Like Fractions

If the denominators of two or more fractions are the same, then they are called like fractions. For example 1/6, 2/6, 3/6, 5/6, are known as like fractions. We can perform addition and subtraction of fractions only on like fractions. In some cases, we need to convert unlike fractions to like fractions to add or subtract.

Unlike Fractions

If the denominators of two or more fractions are different, then the fractions are termed as unlike fractions. For example 1/2, 1/3, 2/5, 3/6, etc. If the fractions are unlike, while adding or subtracting the fractions, we convert them to like fractions.

Equivalent Fractions

Equivalent fractions are the fractions that have different numerators and different denominators but are equal to the same value when simplified or reduced. For example, 2/4, 3/6, 4/8 are all equivalent fractions because they all get reduced to 1/2.

Apart from these, there are fractions with 1 as the numerator. Let us read about them.

Unit Fractions

Units fractions are those fractions in which the numerator is 1 and the denominator is a positive integer. For example, 1/3, 1/8, 1/19, 1/23, and so on are called unit fractions.

Improper Fraction to Mixed Fraction

To convert improper fractions to mixed fractions, we need to divide the numerator by the denominator. Then, we write it in the mixed number form by placing the quotient as the whole number, the remainder as the numerator and the divisor as the denominator. Let us go through the following example to understand this better. Let the improper fraction be 12/5. To convert it into a mixed fraction, we follow these steps:

- Step 1: Divide 12 by 5.
- Step 2: On dividing, we obtain the quotient as 2 and the remainder as 2.
- Step 3: The quotient becomes the whole number part and the remainder 2 becomes the new numerator, while the denominator remains the same.
- Step 4: Thus, the improper fraction, 12/5 is written as a mixed fraction and is represented as 225225

Mixed Fraction to Improper Fraction

A mixed fraction is a mixture of a whole number and a proper fraction. In order to convert a mixed fraction to an improper fraction, we need to multiply the denominator with the whole number part and then add the numerator to the product. The resultant will be the new numerator, whereas, the denominator remains the same. Let us go through the following example to understand this better. Let the mixed fraction be 735735. To convert this into an improper fraction, we follow these steps:

- Step 1: Multiply the whole number 7 with the denominator 5. So, we get 7 × 5 = 35
- Step 2: Add the product with the numerator: 35 + 3 = 38

- Step 3: Express it as a fraction with the denominator 5, that is, 38/5

Tips on Types of Fractions

Given below are a few important points related to the different types of fractions:

- The value of an improper fraction is always greater than 1.
- The value of a proper fraction is always less than 1.
- A mixed fraction is the combination of a whole number and a fraction.
- A mixed fraction can be converted into an improper fraction and vice versa. For example, 212212 = 5/2.

Converting a mixed number to an improper fraction and vice versa: converting an improper fraction to a mixed number

Improper Fraction to Mixed Number

In order to convert an **improper fraction to a mixed number**, we need to divide the numerator by the denominator. After the division, the mixed number is formed in such a way that the quotient that is obtained becomes the whole number, the remainder becomes the new numerator and the denominator remains the same. Let us learn more about converting an improper fraction to mixed number in this lesson.

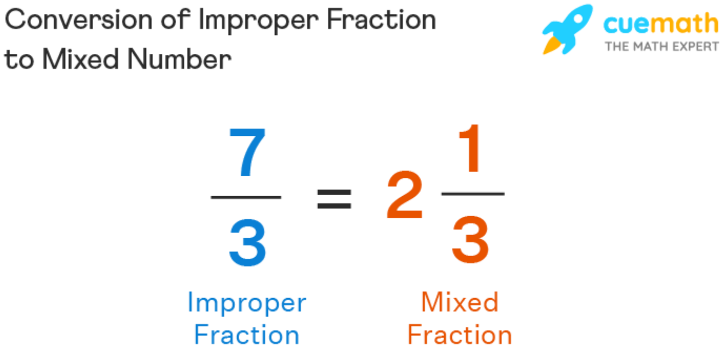
Conversion of Improper Fraction to Mixed Number

An improper fraction is a fraction in which the denominator is always less than the numerator. For example, 9/2 is an improper fraction. A mixed fraction or a mixed number is a combination of a whole number and a proper fraction. For example, 217217 is a mixed number where 2 is the whole number and 1/7 is the proper fraction.

To convert an improper fraction to a mixed number, we need to divide the numerator by the denominator and then find out the remainder and the quotient. Now, the quotient becomes the whole number of the resultant mixed fraction, the remainder becomes the numerator part of the mixed fraction and the denominator part remains the same.

Example: Convert the improper fraction into a mixed number: 7/3

Solution: On dividing 7 by 3, we get 2 as the quotient and 1 as the remainder. Thus, 7/3 will be written as 213213 as a mixed number.



How to Convert Mixed Number to Improper Fraction?

When the numerator (top number) is greater than the denominator (bottom number), that fraction is an improper fraction. An alternative form for an improper fraction is as a mixed number, which is made up of a whole number and a fraction.

For example, you can represent the improper fraction 3/2 as the equivalent mixed number 1-1/2. The mixed number 1-1/2 means 1 + 1/2. To see why 3/2 = 1-1/2, realize that three halves of a cake is the same as one whole cake plus another half. Every improper fraction has an equivalent mixed number, and vice versa.

Sometimes at the beginning of a fraction problem, converting a mixed number to an improper fraction makes the problem easier to solve. Here’s how to make the switch from mixed number to improper fraction:

1. Multiply the whole number by the fraction’s denominator (bottom number).
2. Add the numerator (top number) to the product from Step 1.
3. Place the sum from Step 2 over the original denominator.

Similarly, at the end of some problems, you may need to convert an improper fraction to a mixed number. To do so, simply divide the numerator by the denominator. Then build a mixed number:

- The quotient is the whole number.
- The remainder is the numerator of the fraction.
- The denominator of the fraction stays the same.

As we already know that an improper fraction is a fraction where the numerator is more than the denominator and a mixed number consists of a whole number and a proper fraction. So, while converting a mixed number into an improper fraction we multiply the denominator by the whole number then add the product with the numerator.

Example: Let us convert the mixed number, 517517 to an improper fraction.

Solution: We will multiply the number 7 by 5 and the product is 7 × 5 = 35. To this, we will add the numerator 1, which makes it 35 + 1 = 36. Now, 36 becomes the new numerator and the denominator 7 remains the same. Therefore, 517517 is changed to an improper fraction and is written as 36/7.

Adding Improper Fraction to Mixed number

Adding an improper fraction to a mixed number is simple. We need to convert the mixed number into an improper fraction and then we need to check the denominators of the given fractions which should be the same. In case they are the same, then the numerators can be added while the denominator remains the same. However, if the denominators are different, then they need to be changed to a common denominator. This is done by finding the LCM of the denominators and then the fractions can be added.

When the denominators are same

Example: Add 6/5 and 415415

Solution: We will convert 415415 into an improper fraction

415415 = 21/5

Now, add the numerators: 6/5 and 21/5

(6 + 21)/5 = 27/5. Now, we will finally convert this improper fraction to a mixed number = 525525

When the denominators are not the same

Example: Add 6/5 and 416416

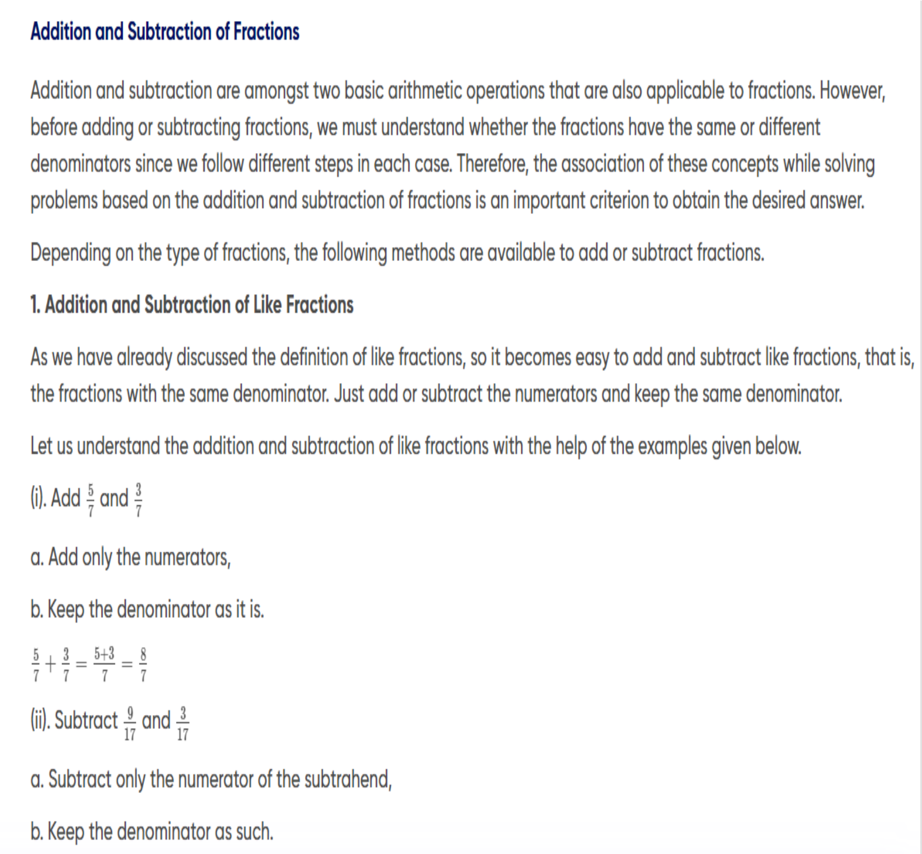
Solution: We will convert 416416 into an improper fraction

416416 = 25/6

Now, add 6/5 and 25/6

Since the denominators are different, we will make their values equal. For this, we need to find the Least Common Multiple (LCM) of the denominators. The LCM of 5 and 6 is 30. Now, we multiply both the fractions with such a number so that the denominators become the same. This means we will multiply 6/5 with 6/6, that is, 6/5 × 6/6 = 36/30. And we will multiply 25/6 with 5/5, that is, 25/6 × 5/5 = 125/30. Now, they can be added and written as (36 + 125)/30 = 161/30. Now, let us convert the improper fraction to a mixed number: 161/30 = 51130

§9. Addition, subtraction and properties of fractional numbers. The concept of approximate number. Reasons for the appearance of approximate numbers. Rounding of approximate numbers



Addition and Subtraction of Fractions

Addition and subtraction are amongst two basic arithmetic operations that are also applicable to fractions. However, before adding or subtracting fractions, we must understand whether the fractions have the same or different denominators since we follow different steps in each case. Therefore, the association of these concepts while solving problems based on the addition and subtraction of fractions is an important criterion to obtain the desired answer.

Depending on the type of fractions, the following methods are available to add or subtract fractions.

1. Addition and Subtraction of Like Fractions

As we have already discussed the definition of like fractions, so it becomes easy to add and subtract like fractions, that is, the fractions with the same denominator. Just add or subtract the numerators and keep the same denominator.

Let us understand the addition and subtraction of like fractions with the help of the examples given below.

(i). Add $\frac{5}{7}$ and $\frac{3}{7}$

a. Add only the numerators,

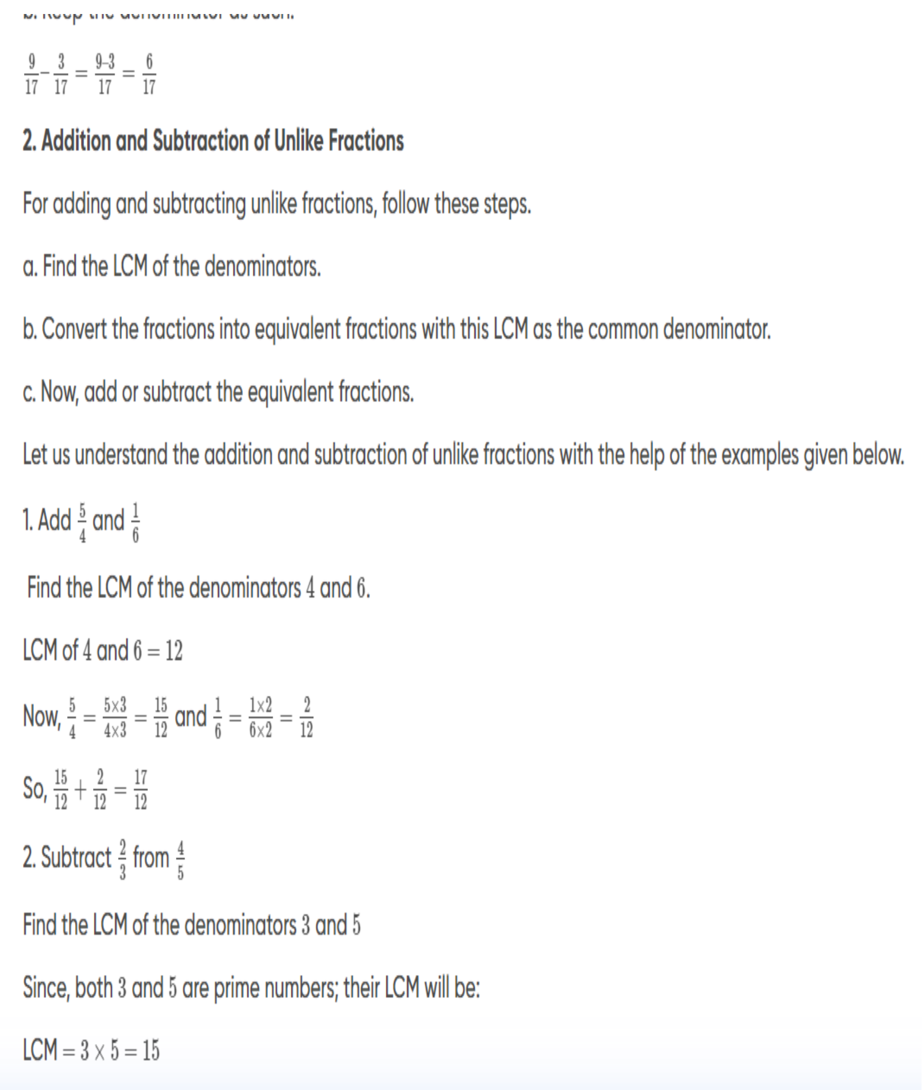
b. Keep the denominator as it is.

$\frac{5}{7}+\frac{3}{7}=\frac{5+3}{7}=\frac{8}{7}$

(ii). Subtract $\frac{9}{17}$ and $\frac{3}{17}$

a. Subtract only the numerator of the subtrahend,

b. Keep the denominator as such.



b. Keep the denominator as such.

$\frac{9}{17} - \frac{3}{17} = \frac{9-3}{17} = \frac{6}{17}$

2. Addition and Subtraction of Unlike Fractions

For adding and subtracting unlike fractions, follow these steps.

a. Find the LCM of the denominators.

b. Convert the fractions into equivalent fractions with this LCM as the common denominator.

c. Now, add or subtract the equivalent fractions.

Let us understand the addition and subtraction of unlike fractions with the help of the examples given below.

1. Add $\frac{5}{4}$ and $\frac{1}{6}$

Find the LCM of the denominators 4 and 6.

LCM of 4 and $6 = 12$

Now, $\frac{5}{4} = \frac{5 \times 3}{4 \times 3} = \frac{15}{12}$ and $\frac{1}{6} = \frac{1 \times 2}{6 \times 2} = \frac{2}{12}$

So, $\frac{15}{12} + \frac{2}{12} = \frac{17}{12}$

2. Subtract $\frac{2}{3}$ from $\frac{4}{5}$

Find the LCM of the denominators 3 and 5

Since, both 3 and 5 are prime numbers; their LCM will be:

LCM $= 3 \times 5 = 15$

Now, $\frac{2}{3} = \frac{2\times5}{3\times5} = \frac{10}{15}$ and $\frac{4}{5} = \frac{4\times3}{5\times3} = \frac{12}{15}$

So, $\frac{12}{15} - \frac{10}{15} = \frac{2}{15}$

3. Addition and Subtraction of Mixed Fractions

We have provided the steps one needs to follow in order to add and subtract the mixed fractions below:

1. Convert the mixed fraction into an improper fraction.
2. Convert the improper fraction obtained into like fractions.
3. Keep the denominator the same, add or subtract the numerators of the equivalent fractions and obtain a single fraction.
4. Reduce the fraction into its lowest terms, if required, and then convert it again into a mixed fraction.

Solve $1\frac{3}{7} - \frac{1}{6}$

$= \frac{10}{7} - \frac{1}{6}$

$= \frac{10\times6}{7\times6} - \frac{1\times7}{6\times7} = \frac{60}{42} - \frac{7}{42}$

$= \frac{60-7}{42} = \frac{53}{42} = 1\frac{11}{42}$

Addition of Fractions

Case 1: If the denominators are the same for given fractions, we can directly add the numerators of fractions. And these fractions with the same denominators are called “Like Fractions.”

Example:

27+37=5727+37=57

Case 2: If the denominators are different, the fractions are called, Unlike fractions. In this case, we can not add the numerators directly. First, we need to make the fractions like fractions (Same denominators). Conversion of Unlike fractions to Like Fractions can be done using one of the two methods shown below:

If the denominators are not the same for given fractions

Step 1: Find out the LCM of both denominators.

Step 2: Using the LCM, we could make the Unlike fractions to Like fractions, and then the numerators are simplified.

Example:

23+5723+57

In this example LCM =21=21

2×73×7+5×37×3=1421+1521=14+1521=29212×73×7+5×37×3=1421+1521=14+1521=2921

Method 2

Step 1: In this method, we cross multiply the numerator of the first fraction to the denominator of the second fraction and numerator of the second fraction to the denominator of the first fraction.

Step 2: Then multiply both denominators.

In either way, the answer will be the same.

Example:

23+57(2×7)+(5×3)(3×7)=292123+57(2×7)+(5×3)(3×7)=2921

Case 3: The addition of mixed fractions can be done in two ways.

Method 1:

Step 1: Add the Wholes and fractions separately as shown.

Step 2: If the fractions to be added are Unlike fractions, add them by finding the LCM of the denominators or by using the cross multiplication method, as shown in Case 2.

Example:

123+314=(1+3)(23+14)=41112123+314=(1+3)(23+14)=4 1112

Method 2:

In this method,

Step 1: Convert mixed fraction into an improper fraction as shown in the example below.

Step 2: Add them by following the above-discussed methods.

Example:

123+314123=(3×1)+23=53314=(3×4)+14=134123+31412 3=(3×1)+23=53314=(3×4)+14=134

Add 53,13453,134

53+134=20+3912=5912=4111253+134=20+3912=5912=4 1112

Subtractions of Fractions

The subtraction of fractions is similar to the addition of fractions as in the above examples. Instead of addition, we need to subtract at the relevant places.

Case 1: In the case of Like fractions (fractions with the same denominators), we can directly subtract the numerators of fractions.

Example:

107−27=57107−27=57

Case 2: Unlike fractions, we use two different methods, either by taking the LCM of the denominators and making it into a Like fraction and then subtracting it or by cross multiplying. Both the methods are shown with examples below:

Method 1:

Step 1: Find out the LCM of both denominators.

Step 2: Using the LCM, we could make the Unlike fractions to Like fractions, and then the numerators are simplified.

Example:

103−57103−57

In this example LCM =21=21

10×73×7−5×37×3=7021−1521=552110×73×7−5×37×3=7021−1521=5521

Method 2:

Step 1: Cross multiply the numerator of the first fraction to the denominator of the second fraction and numerator of the second fraction to the denominator of the first fraction.

Step 2: Then Multiply both denominators.

Example:

103−57(10×7)−(5×3)(3×7)=5521103−57(10×7)−(5×3)(3×7)=5521

Case 3: Subtraction of mixed fractions

Method 1:

In the case of the subtraction of mixed fractions, first, we have to separate the whole part and the fractional part. And then, we can subtract them as per the method discussed above if denominators are different.

Example:

323−114=(3−1)(23−14)=2512323−114=(3−1)(23−14)=25 12

Method 2:

In this method of subtraction of mixed fractions, we have to convert a mixed fraction into an improper fraction.

Example:

323−114=(3×3)+23−(1×4)+14=113−54=(11×4)−(5×3)12=3912=2512323−114=(3×3)+23−(1×4)+14=113−54=(11×4)−(5×3) 12=3912=2512

Multiplication of Fractions

The multiplication of fractions is quite simple. We have to multiply numerators of given fractions and then multiply denominators of the given fractions. If required we could convert the fraction into its simplest form or the reduced form.

Examples:

1. 45×79=4×75×9=284545×79=4×75×9=2845
2. 34×8=3×84=244=634×8=3×84=244=6
3. 123×335=53×185=5×183×5=9015=183123×335=5 3×185=5×183×5=9015=183 (hence 183183 is the reduced form of 905905)

Division of Fractions

The division of fractions includes the following steps

Step 1: Here we need to include the concept of reciprocal of fractions which means the inverse of the given fraction.

Example:

1. Reciprocal of 2323 is 3232
2. A reciprocal of 5 is 1515

Step 2: Multiply the first fraction with the reciprocal of the first fraction.

Examples of the division of fractions

1. 35÷57=35×75=212535÷57=35×75=2125
2. 5÷57=5×75=355=75÷57=5×75=355=7 (Simplifed form)
3. 45÷6=45×16=21545÷6=45×16=215 (Reduced form)

Comparison of Fractions

The larger the numerator the larger the fraction, and the larger the denominator the smaller the fraction.

If the denominators are the same, the fraction with a larger numerator is larger and if the numerators are the same, the fraction with the larger denominator is smaller.

Case 1: When denominators are the same but numerators are different.

In this example, the fraction having a greater numerator is the greater fraction.

Example:

Compare 6868 and 7878

68<7868<78

Case 2: When numerators are the same but denominators are different.

In this example, the fraction having a lesser denominator is the greater fraction.

Example:

Compare 139139 and 13101310

139>1310139>1310

Case 3: When numerators and denominators of the given fractions are different, we have to make the fractions into like fractions by finding the LCM of denominators. The like fractions are then compared as shown above.

Example:

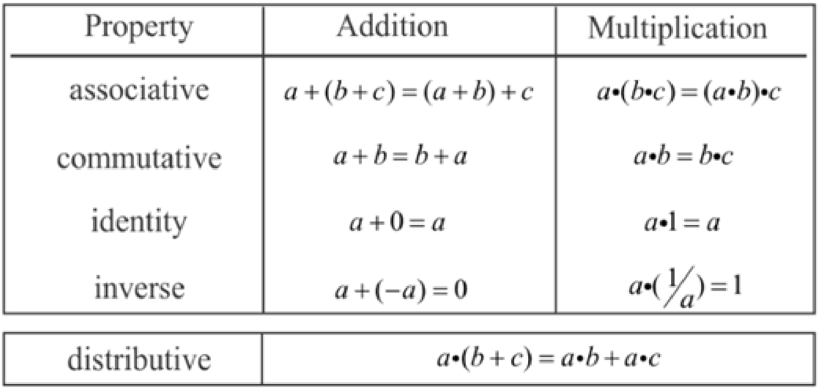
Compare 2525 and 4747

LCM of 5 and 7 is 35

2×75×7=1435 and 4×57×5=20351435<20352×75×7=1435 and 4×57×5=20351435<2035

Properties of Fractions

There are different properties of fractions over multiplication and addition and they are listed in the below table.



Property	Addition	Multiplication
associative	$a+(b+c)=(a+b)+c$	$a\bullet(b\bullet c)=(a\bullet b)\bullet c$
commutative	$a+b=b+a$	$a\bullet b=b\bullet c$
identity	$a+0=a$	$a\bullet 1=a$
inverse	$a+(-a)=0$	$a\bullet(\frac{1}{a})=1$

distributive	$a\bullet(b+c)=a\bullet b+a\bullet c$

Each property is explained with examples below.

Examples of properties of fractions

1. Associative Property

Similar to Numbers and Integers, associative property gets applied in fractions too, over Addition and Multiplication.

Associative Property of Addition

Example:

12+(13+14)=(12+13)+1412+(13+14)=(12+13)+14

In this example, the answer will be on both the LHS and RHS side. Hence we can say that fractions follow the associative property of addition.

Associative Property of Multiplication

Example:

25×(14×32)=(25×14)×3225×(14×32)=(25×14)×32

In this example, the answer will be 320320 on both the LHS and RHS side.

Hence we can say that fractions follow the associative property of addition.

2. Commutative Property

Commutative Property of Addition

Example:

13+14=14+1313+14=14+13

In this example, the answer is 712712 on both the LHS and RHS side. Hence we can say that fractions follow the associative property of addition. This property is followed by all types of fractions.

Commutative Property of Multiplication

Example:

14×32=14×3214×32=14×32

In this example, the answer is 3838 on both the LHS and RHS side. Hence we can say that fractions follow the associative property of addition. This property is followed by all types of fractions.

3. Identity Property

Identity Property of Addition

Example:

45+0=4545+0=45

The addition of zero to any fraction gives the fraction itself.

Identity Property of Multiplication

Example:

45×0=045×0=0

Any fraction multiplied by zero gives zero.

4. Inverse Property

Identity Property of Addition

Example:

35+(−35)=035+(−35)=0

The identity property of addition gives 0.

Identity Property of Multiplication

Example:

35×135=135×135=1

Identity property of addition gives 1.

5. Distributive Property

Example:

35(23+15)=(35×23)+(35×15)35(23+15)=(35×23)+(35×15)

In this example, the answer is 13251325 on both the LHS and RHS side. Hence we can say that fractions follow the distributive property of addition.

Conversion of Fraction to Decimal numbers

Usually, this conversion is done using any of the two methods:

1. Long division method - Dividing the numerator with the denominator.
2. By converting the denominator into a number which is powers of 10. Like 10, 100, 1000, etc. If it is converted it is easy

for us to find the decimal number when the fraction has powers of 10 in the denominator.

Examples

1. 512=0.4166512=0.4166
2. 125215=0.581125215=0.581
3. 58=5×1258×125=6251000=0.62558=5×1258×125 =6251000=0.625
4. 34=3×254×25=75100=0.7534=3×254×25=75100= 0.75

Practice Questions

Question 1: The decimal representation of 235235 is ______

(A) 2.5 (B) 2.6 (C) 2.55 (D) 2.50

Question 2: 130.02 + 113.26 – 27.1 – 17.4 = ?

(A) 180.78 (B) 178.78 (C) 286.78 (D) 198.7

Question 3: The product of two decimals is 33.655. If one of them is 1.27, the other number is _____________.

(A) 27.5 (B) 26.5 (C) 27.25 (D) 25.75

Question 4: The value of "[(−234)−(−134)]+[(−234)−(−134)]+…[(−234)−(−134)]+[(−234) −(−134)]+…" up to 30 times is __________.

(A) –1 (B) 1 (C) 30 (D) –30

Question 5: Simplify: 222−[13{42+(56−8– +9)}+108]222−[13{42+(56−8_+9)}+108]

(A) 87 (B) 195 (C) 89 (D) 159

Question 6: 123.45×67.890×1.032123.45×67.890×1.032 is the same as________.

(A) 1234.5×6.789×103201
234.5×6.789×10320

(B) 1234.5×67.89×103.21
234.5×67.89×103.2

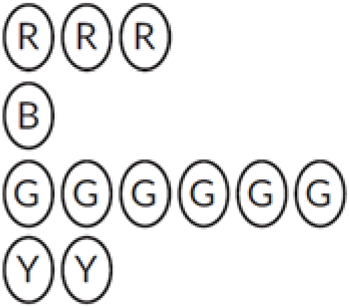
(C) 12.345×6789.0×10.321
2.345×6789.0×10.32

(D) 1234.5×67.89×0.1032
1234.5×67.89×0.1032

Question 7: A farmer has 384 animals, out of which one-third is cows and one-fourth of the cows are dairy cows. How many dairy cows does he have?

(A) 160 (B) 96 (C) 32 (D) 128

Question 8: Gopal has a bag that contains 3 red, 1 blue, 6 green, and 2 yellow marbles. What fractional part of the bag of marbles is red?



(A) 112112 (B) 212212 (C) 312312 (D) 6126

Answers with explanation

1.

Solution: 235=(2×5)+35=135=2.6235=(2×5)+35=135=2.6

Ans: B

2. Solution: 130.02+113.26–27.1–17.4=198.78130.02+113.26–27.1–17.4=198.78

Ans: D

3. Solution: Let the other number be x.

1.27×X=33.655X=33.6551.27=26.51.27×X=33.655X=33.6551.27=26.5

Therefore, the other number is 26.5.

Ans: B

4. Solution: [(−234)−(−134)]+[(−234)−(−134)]+…………+30[(−234)−(−134)]+[(−234)−(−134)]+…………+30 **times**

[(−234)−(−134)]=−1−1×30=−30[(−234)−(−134)]=−1−1×30=−30

Ans: D

5.Solution: 222−[13{42+(56−8+9)}+108]222−[13{42+(56−8_+9)}+108]

In this example we have to solve the innermost bracket first

222[13{81}+108]=222−135=87222[13{81}+108]=222−135=87

Ans: A

6.Solution: 123.45×67.890×1.032=1234.5×67.89×0.1032123.45×67.890×1.032=1234.5×67.89×0.1032

L.H.S.=123.45×67.890×1.032=8649.213156=123.45×67.890×1.032=8649.213156

R.H.S.=1234.5×67.89×0.1032=8649.213156=1234.5×67.89×0.1032=8649.213156

Among all the given options only option D is giving the same that is 8649.213156

Ans: D

7.Solution: 13×384=12813×384=128 is cows and 14×128=3214×128=32 Dairy cows

Ans: C

8. Solution: A bag that contains 3 red, 1 blue, 6 green, and 2 yellow marbles

Total number of balls =12=12

The fraction of red marbles =no of balls in red colortotal number of balls in bag=34=no of balls in red colortotal number of balls in bag=34

Ans: C

Learning outcomes

- Identify the number of shaded parts and the number of equals parts in a shape (circle, rectangle).
- Identify a fraction by comparing the number of shaded parts to the total number of parts.
- Write a fraction using mathematical notation and using words.
- Recognize, the value of a fraction is dependent on the number of shaded parts and not on the position of the shaded parts.
- Understanding how to solve addition, subtraction, multiplication, and division of fractions.

Reasons for the appearance of approximate numbers. Rounding of approximate numbers

Sometimes, you may find it helpful to know the approximate answer to a calculation.You may be in a shop and want to know broadly what you’re going to have to pay.You may need to know roughly how much money you need to meet a couple of bills.

You may also want to know roughly what the right answer to a more complicated calculation is likely to be, to check that your detailed work is correct. Whatever your precise need, you want to know how to estimate or approximate the right answer.

Rounding

One very simple form of estimation is rounding. Rounding is often the key skill you need to quickly estimate a number. This is where you make a long number simpler by ‘rounding’, or expressing in terms of the nearest unit, ten, hundred, tenth, or a certain number of decimal places.

For example, 1,654 to the nearest thousand is 2,000. To the nearest 100 it is 1,700. To the nearest ten it is 1,650.

The way it works is straightforward: you look at the number one place to the right of the level that you are rounding to and see whether it is closer to 0 or 10. In practice, this means that if you’ve been asked to round to the nearest 10, you look at the units. If you are rounding to three decimal places, you look at the fourth decimal place (the fourth number to the right of the decimal point) and so on. If that number is 5 or over, you round up to the next number, and if it is 4 or under, you round down.

Rp or Round **Rounding: Worked Examples**

Example

Express 156 to the nearest 10

In this example you look at the tens and units. The hundreds will not change. You need to decide whether 56 will be rounded up to 60 or down to 50.

Looking at the units, you know that 6 is more than 5, so you round up.

The answer is 160.

Example 2

Express 0.4563948 to three decimal places.

As you're working to three decimal places, the answer will start 0.45 and you need to determine the third number after the decimal point

To work out whether the third number is 6 or 7, you need to look at the fourth number, which is 3. As 3 is less than 5, you round down.

The answer therefore is 0.456.

You can use the technique of rounding to start estimating the answer to more complex problems.

Estimation

Estimating can be considered as ‘slightly better than an educated guess’. If a guess is totally random, an educated guess might be a bit closer.

Estimation, or approximation, should give you an answer which is broadly correct, say to the nearest 10 or 100, if you are working with bigger numbers.

Probably the simplest way to estimate is to round all the numbers that you are working with to the nearest 10 (or 100, if you are working in thousands at the time) and then do the necessary calculation.

For example, if you are estimating how much you will have to pay, first round each amount up or down to the nearest unit of currency, pound, dollar, euro etc. or even to the nearest 10 units (£10, $10, €10), and then add your rounded amounts together.

Many stores like to give prices ending in .09 and especially 0.99. The reason for this is that a shirt that costs 24.99 'sounds' cheaper than one that costs 25.00. When shopping for numerous items it can be useful to keep a running tally, an estimate of the total cost, by rounding items to the nearest currency unit, £, $, € etc.

If you are trying to work out how much carpet you will need, round the length of each wall up to the nearest metre or half-metre if the calculation remains simple, and multiply them together to get the area.g!

Example 1

You want to buy carpet for two rooms. The first is 3.2m by 2.7m. The second is smaller, 1.16m by 2.5m. How much carpet do you need to buy to be sure of having enough for both rooms?

The first room is approximately 3m by 3m, which is 9m^2.

The second is just over 1m by 2.5m. Strictly speaking, you would round this to 1m by 2.5m, or 2.5m^2.

In total, then, that's 11.5m^2. It's hard to buy carpet in anything except whole m^2, so you'll need to round up to 12m^2. In each case, you have rounded up one of the numbers by more than you have rounded the other one down, so you're probably fine.

A quick check with a calculator will, indeed, confirm that you need exactly 11.54m^2. 12m^2 will be plenty.

Example 2

You've decided to add another room to the carpet buying. The last room is 3.9m by 2.2m. How much carpet do I need for all three rooms?

3.9m is rounded up to 4m. 2.2m rounds down to 2m.

2 × 4 is 8m^2, which gives a total, for all three rooms of 20m^2.

However, in rounding down to 2m, you have taken out 0.2m. In rounding up to 4m, you have only added 0.1m.

You may not order quite enough carpet although you might get away with it because you rounded up to $12m^2$ for the first two rooms.However, to be absolutely sure, you probably want to round 2.2m up, to 2.5m.Multiply 2.5 by 4 to get $10m^2$. This means you need $22m^2$ of carpet for all three rooms.A quick check with a calculator will confirm that $20m^2$ is not quite enough: You need $20.9m^2$ exactly.

Need a refresher on how to calculate area? See our page **Calculating Area** for help.

Estimated Time of Arrival (ETA)

Estimated time of arrival is used frequently used when travelling. Trains, buses, planes, ships and in-car satellite navigation (sat-nav) all use ETA.

The ETA is based on distance and speed of travel, it is 'estimated' because it cannot take into account changes to speed during the journey. Your flight may arrive early because of favourable tail winds. Your road trip may take longer than expected because of traffic.

The ETA is usually calculated by a computer and can change during your trip. As you near your destination, more data becomes available so the estimated time that you will arrive becomes more accurate.

A Special Case: Estimating for Work

You will almost certainly come across 'estimates' for work to be done, whether from a builder, plumber, mechanic or other tradesperson.

In this case, the tradesperson concerned has probably estimated how much time they are likely to take to do the work, multiplied it by their hourly or daily rate, and perhaps added additional charges for materials or a call-out.

They may also have added a ‘contingency’ for extra work needed, which is likely to be 10 or 20%, and will mean that you are not unpleasantly surprised by the bill if they find something unexpected that needs fixing.

An ‘estimate’ is not legally binding. It is just what it says: an estimate.

However, a ‘quote’ or ‘quotation’ for work done is legally binding on cost, provided that the work done is what was quoted for. However, if you have asked for extra work: ‘just add that bit’ or ‘do that while you’re here’, don’t be surprised if the bill is larger than you were expecting.

A Useful Skill

You may be wondering why you’d ever use estimation when you have a calculator on your phone.

The ability to estimate will mean that you will know if the answer you get from the calculator is not right, and do it again.

Rounding of approximate numbers

Rounding Numbers

Rounding Numbers means adjusting the digits of a number in such a way that it gives an approximate value. This value is an easier representation of the given number. For example, the population of a town could be easily expressed as 700,000 rather than 698,869. Rounding numbers makes calculations simpler, resulting in a figure that is easy to remember. However, rounding of numbers is done only for those numbers where the exact value does not hold that much importance.

Let us learn more about rounding numbers, to get a better idea of how to round off a number to the nearest ten, hundred, thousand, and so on.

What is Rounding Numbers?

Rounding a number means the process of making a number simpler such that its value remains close to what it was. The result obtained after rounding off a number is less accurate, but easier to use. While rounding a number, we consider the place value of digits in a number.

Let us understand the concept of rounding through an example. Susan covered a distance of 2.05 miles, which she noted as approximately 2 miles. How did she estimate the approximate value of the distance she covered? Why didn't she record her distance as 3 miles? Noting an approximate and simpler value for a given number helps to make the analysis and calculations easier while using that number. Here, Susan noted an easier value to keep a record of the distance travelled by her.

Numbers can be rounded to different digits, like, they can be rounded to the nearest ten, hundred, thousand, and so on. For example, 541 rounded to the nearest hundred is 500 because 541 is much closer to 500 than 600. While rounding a number, we need to know the answer to the question,' What are we rounding the number to?' Suppose we need to round the number 7456. When 7456 is rounded to the nearest ten, it becomes 7460, but when 7456 is rounded to the nearest thousand, then it becomes 7000. A few examples like these are given below.

Rounding Numbers to the Nearest Ten

Rounding numbers to the nearest ten means we need to check the digit to the right of the tens place, that is the ones place.

For example, when we round the number 7486 to the nearest ten, it becomes 7490.

Rounding Numbers to the Nearest Hundred

Rounding numbers to the nearest hundred means we need to check the digit to the right of the hundreds place, that is, the tens place. For example, when 7456 is rounded to the nearest hundred, it becomes 7500.

Round-Up and Round-Down

While rounding is a generic term, we usually use the terms, 'round up' or 'round down' to specify if the number has increased or decreased after rounding. When the rounded number is increased, then the given number is said to be rounded up, whereas, if the rounded number is decreased, then it is said to be rounded down.

Rules for Rounding Numbers

How do we decide which value is more appropriate between different approximated values of a number? Should we choose a number greater than the given number or go with the smaller one?

There are some basic rules that need to be followed for rounding numbers.

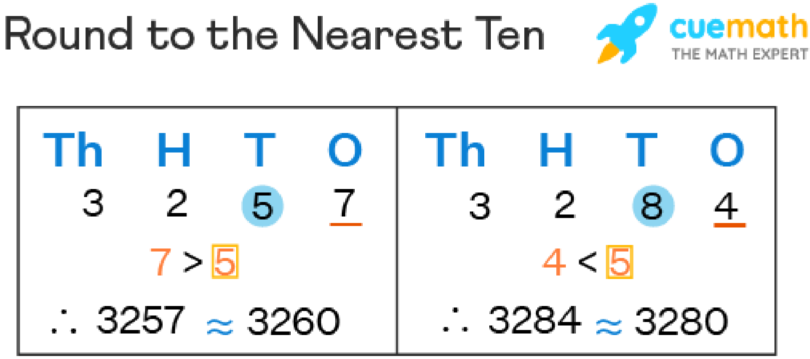
- We first need to know what our rounding digit is. This digit is the one that will ultimately be affected.
- After this, we need to check the digit to the right of this place which will decide the fate of the rounding digit.
- If the digit to the right is less than 5, we do not change the rounding digit. However, all the digits to the right of the rounding digit are changed to 0.
- If the digit to the right is 5 or more than 5, we increase the rounding digit by 1, and all the digits to the right of the rounding digit are changed to 0.

Example:

a.) If the bill at a furniture store comes to $3257, what is the rounded value of the amount to the nearest ten?

b.) If the bill comes to $3284, what would be the rounded value of this amount to the nearest ten?

Solution:



a.) $3257 needs to be rounded to the nearest ten. So, let us mark the digit in the tens place, which is 5. Now, let us check the number to the right, which is 7 in this case. Since 7 is more than 5, we will replace 5 with 6, and all the digits to the right will become 0. So, $3257 is rounded to $3260.

b.) Here, $3284 needs to be rounded to the nearest ten. So, let us mark the digit in the tens place, which is 8. Now, let us check the number to the right, which is 4 in this case. Since 4 is less than 5, 8 will remain unchanged and the remaining digits to the right will change to 0. So, $3284 is rounded to $3280.

How to Round Off Whole Numbers?

Whole numbers are rounded off by following the same rules mentioned above. Let us apply the rules with the help of an example.

Example: Round 7234 to the nearest hundred.

- **Step 1:** Mark the place value up to which the number needs to be rounded. Here, 7234 needs to be rounded to the nearest hundred. So, we mark 2 which is in the hundreds place.
- **Step 2:** Observe and underline the digit to the right of 2, that is the tens place. Here, it is 3, so we will mark it as: 7**2**34
- **Step 3:** Compare the underlined digit with 5.
- **Step 4:** If it is less than 5, all the digits towards its right including it will be replaced by 0 while the digit in the hundreds place (2) will remain unchanged. Therefore, 7234 will be rounded to 7200.

Note: If the number to the right of 2 was 5 or greater than 5, then all the digits to the right of 2 would become 0, and 2 would be increased by 1 changing it to 3. For example, if the given number was 7268, then it would be rounded up to 7300 (to the nearest hundred).

How to Round Off Fractions?

Fractions are numerical values that represent a part of a whole. They are written in the form of (p/q), where q is not equal to zero. A simple rule to round fractions to the nearest whole number is to compare proper fractions to 1/2. In case if it is an improper fraction, we need to change it to a mixed fraction and then compare the fractional part with 1/2. We round up the fraction if it is equal to or greater than 1/2, and we round down if it is less than 1/2. Let us understand rounding off fractions to the nearest whole number using the following example.

Example: Round the given fractions to the nearest whole number:

a.) 3/4

b.) 625625

c.) 21/5

Solution:

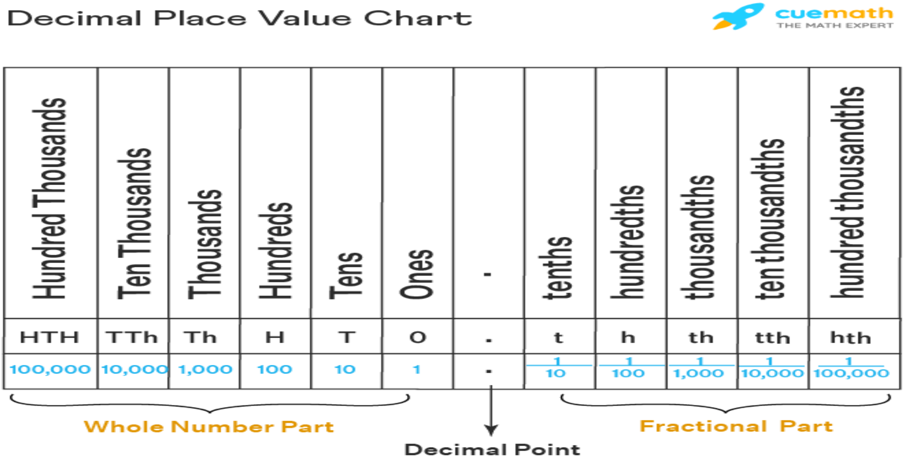
a.) We can round off a proper fraction to the nearest whole number by following the simple rule of comparing it with 1/2. Since 3/4 is greater than 1/2, it will be rounded off to 1.

b.) 625625 is a mixed fraction. Here, we will keep the whole number part aside and compare the fractional part with 1/2. So, keeping 6 aside, we will check if 2/5 is greater than or less than 1/2. Since 2/5 is less than 1/2, we will round the given mixed fraction to 6.

c.) 21/5 is an improper fraction, so we will convert it to a mixed fraction. This will make it 415415. Now, we will keep the whole number part aside and compare the fractional part with 1/2. So, keeping 4 aside, we will check if 1/5 is greater than or less than 1/2. Since 1/5 is less than 1/2, we will round the given fraction to 4.

How to Round Off Decimal Numbers?

A decimal number is a combination of a whole number part and a fractional part separated by a decimal point. Rounding decimal numbers works in the same way as we round whole numbers although we need to know the decimal place values of all the digits in the given number. This refers to the digits given before the decimal point as well as the digits given after the decimal point. Observe the decimal place value chart to understand this better.



Decimal Place Value Chart

cuemath THE MATH EXPERT

Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones	.	tenths	hundredths	thousandths	ten thousandths	hundred thousandths
HTH	TTh	Th	H	T	O	.	t	h	th	tth	hth
100,000	10,000	1,000	100	10	1	.	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$	$\frac{1}{10,000}$	$\frac{1}{100,000}$

Whole Number Part

Fractional Part

Decimal Point

We usually round decimal numbers to the nearest tenths, hundredths, thousandths, and so on, which represent the place values after the decimal point. However, sometimes we even need to round a decimal to the nearest whole number. In this case, we check the tenths digit. If it is equal to or more than 5, then the given number is rounded up, and if the tenths digit is less than 5 then the given number is rounded down.

Example 1: Round 5.62 to the nearest whole number.

Solution: Since we need to round this decimal to the nearest whole number, we will check the tenths digit. In this case, the tenths digit is 6, which is more than 5. So, the number will be rounded to 6. In other words, $5.62 \approx 6$

In the other cases, where we need to round decimals to the nearest tenths, hundredths, or thousandths, we need to remember the simple rule of marking the number up to which we are rounding and checking the number to its right. For example, when we round decimal numbers to the nearest hundredths, we need to check the thousandths place. Similarly, if we need to round decimal numbers to the nearest tenths, we need to check the hundredths place. If the number to be checked is less than 5, then the rounding number remains unchanged, and the following digits are replaced with 0. Whereas, if the number to be checked is 5 or more than 5, then the rounding number is increased by 1 and the following digits are changed to 0. Let us understand this with an example.

Example 2: Round 0.439 to the nearest hundredths.

Solution: In this case, the digit to the right of the hundredths place, that is, the thousandths place is 9, which is more than 5. So, we will add 1 to the digit in the hundredths place, that is, 3 + 1 = 4, and write 0 in the digits to the right. So, 0.439 will be rounded to 0.44

Note: It should be noted that when we round a decimal number to the nearest hundredth, the round decimal fraction is said to be correct to two places of decimal. In other words, if we are asked to round off a number to two decimal places, it means we need to round it to the nearest hundredths. Similarly, when we are asked to round off a number to one decimal place, it means we need to round it to the nearest tenths**.**

Tips on Rounding Numbers:

The following tips are helpful in solving questions related to rounding numbers.

- While rounding numbers, we always need to check the digit to the right of the rounding number. If it is less than 5, the rounding number remains the same and the following digits are changed to 0. If the digit to the right is equal to or more than 5, we increase the rounding digit by 1 and the following numbers are changed to 0.
- When we round decimal numbers, we usually come across terms like round to the nearest tenths, hundredths, thousandths, and so on.

- When we are asked to round off a number to one decimal place, it means we need to round it to the nearest tenths, similarly, when we are asked to round off a number up to two decimal places, it means we need to round it to the nearest hundredths.
- When the rounded number is increased, then the given number is said to be rounded up, whereas, if the rounded number is decreased, then it is said to be rounded down.

§10. Concept of the task. Simple and complex tasks. Classification of account tasks. Stages and methods of problem solving

Maths is essential in everyday life. When leaving primary school, children need to have sound numeracy and the ability to reason and enquire mathematically. Curiosity and pleasure in maths drive science, technology and engineering, and are essential to understanding and development in these areas. For everyone, numeracy is critical in securing employment and managing money.

To be a mathematician at Wardley means that you will have:

- An understanding of the important concepts and an ability to make connections within mathematics.
- A broad range of skills in using and applying mathematics.
- Fluent knowledge and recall of number facts and the number system.
- The ability to show initiative in solving problems in a wide range of contexts, including the new or unusual.
- The ability to think independently and to persevere when faced with challenges, showing a confidence of success.

- The ability to embrace the value of learning from mistakes and false starts.
- The ability to reason, generalise and make sense of solutions.
- Fluency in performing written and mental calculations and mathematical techniques.
- A wide range of mathematical vocabulary.
- A commitment to and passion for the subject.

Subject Implementation

To see what maths topics and skills are taught in each year group click on the maths teaching cycle tab.

At Wardley we follow the 2016 National Curriculum, enriching and structuring it to meet the needs of our students.

The focus of our maths curriculum is on teaching to mastery by ensuring a child thoroughly comprehends a topic before moving on. Ideas are revisited in a spiral as children progress through the school, each time at a higher level. We empathises problem-solving and pupils using their core competencies to develop a relational understanding of mathematical concepts. To assist in this we use the Maths – No problem! scheme of work. This is a Singapore method of teaching mathematics that develops pupils' mathematical ability and confidence. The features of our maths teaching therefore include:

- Emphasis on problem solving and comprehension, allowing children to relate what they learn and to connect knowledge
- Careful scaffolding of core competencies of:
- visualisation, as a platform for comprehension
- mental strategies, to develop decision making abilities
- pattern recognition, to support the ability to make connections and generalise

- Emphasis on the foundations for learning and not on the content itself so children learn to think mathematically as opposed to merely reciting formulas or procedures

It is based upon nine units which the children continually re-visit within a spiral curriculum. They are: Number and Place Value; Addition and Subtraction; Multiplication and Division; Fractions; Decimals and Percentages; Statistics; Time and Money; Other Measures; Shape; and Position and Direction. Through these units we develop the following key mathematical threshold concepts:

- To know and use numbers
- To add and subtract
- To multiply and divide
- To use fractions
- To understand the properties of shapes
- To describe position, direction and movement
- To use measures
- To use statistics
- To use algebra

The children are assessed by the teacher during each unit against the age related expectations for these key threshold concepts. A termly assessment is made for each threshold concept on the school's tracking system.

How do we teach maths?

We use the Maths – No problem! Scheme which uses the ideas from Singapore maths as a support for our maths teaching across the school. Ofsted, the National Centre for Teaching Mathematics (NCETM), the Department for Education, and the

National Curriculum Review Committee have all emphasised the pedagogy and heuristics used by Singapore.

The teaching focuses on three modes of representation of mathematical ideas: the enactive, iconic and symbolic modes. Children are introduced to an idea through concrete apparatus (things they can touch and hold) and visual representations (things they can see) to help children to conceptualise and solve problems, allowing them to approach complicated problems, investigate and reason through them. Through this approach children gain confidence as independent learners who are able to use resources and show resilience in solving problems.

The daily maths lessons at Wardley have certain key features:

1. Times tables/Counting

- Learning facts by heart is key to making sustained progress in mathematics: children can use the solution to one problem to help solve others.
- Each year group has counting and times table focuses that are revised, recited and recalled in short sharp bursts every day.

2. Mental maths

- Being able to solve problems in your head helps to develop mathematical confidence, flexibility with numbers, and understanding of place value.
- Children need opportunities to rehearse, revise, and refresh mental maths.
- Each maths lesson contains at least ten minutes of mental arithmetic.
- Different objectives and areas of focus are met in line with the National Curriculum, each week.

3. Maths story

• A maths story is used to engage, motivate and focus the children on what they are learning.

• The maths story contextualises the learning and allows the children to immediately start connecting their learning with prior knowledge.

4. Modelling and practice

• The teacher demonstrates (models) how to solve the problem.

• This is modelled clearly and consistently with regular opportunities for student participation.

• The children all rehearse this core skill. Over the course of a week students will do this in groups, pairs and independently.

5. Problem Solving

- The teacher returns to the maths problem and asks students how to solve it using their new skills.
- Students link their new skills to a problem which either requires them to solve a problem, prove something, test a statement or give an explanation.
- Students often find making links from one problem to another challenging and so through our questioning and consistency we really focus on this skill. This is particularly underlined in our investigations.

Homework

The main homework task for children in years 1 – 6 is to learn their number bonds and times tables:

1. To learn their times tables (number bonds in year 1) and other number facts

• Children are expected to know all of their times tables and the related division facts by the end of the year 4.

- Children can then use their mathematical knowledge to multiply multiples of 10 and decimals.

To help support this the children use Tables Rock Stars for homework tasks from Y1 to Y4. In the upper juniors the children use Mathletics for maths homework tasks.

How can you help at Home?

Ideas about how you can support your children with their maths work may be given in the half-termly newsletters. For further information about the maths curriculum that your child is studying please contact your child's teacher.

Classification of mathematical tasks in primary school

What is Classification in Math?

Classification in math involves categorizing objects and items according to certain characteristics. It is a pre-number learning concept and teaches students about the world around them. When students start completing classification activities, they will begin to learn how to identify different shapes and colors, too.

Classification helps students to improve an array of different math skills, including how to understand the connections and relationships between different objects, sorting and grouping skills, and identification skills among others.

Simple classification math definition

Classification is where objects are systematically organized into groups according to fixed criteria. For example, if a child has a mixture of red and blue items and they were classifying them according to their colors, they would be actively seeking out red objects to classify into their red category and blue objects for their blue category.

Examples of Classification

Three examples of Classification in Math

1. Shapes

Shapes can be classified in all sorts of ways, but during PreK, your students should be learning how to classify shapes according to their various different properties. This could be the name of the shapes, how many sides or corners they have, or whether they are 2D or 3D.

2. Colors

Color identification is another pre-number math skill that students learn during PreK, and classification activities help to reinforce what students have learned about colors. For example, students may be given a pile of colored objects to sort through and be required to put all of the objects that are primary colors (red, blue, and yellow) in one pile, and all of the objects that are secondary colors (purple, green, and orange) in another pile.

3. **Odd and even numbers**

Students can easily classify numbers into the categories of odd or even numbers by looking at the characteristics of the numbers that they are classifying. For example, odd numbers cannot be divided by two and the last digit is always a 1, 3, 5, 7, or 9, and even numbers can always be divided equally by two, with the last digit of an even number always being 0, 2, 4, 6, or 8.

Three examples of classification in everyday life

1. **Eating healthily**

When people make the decision to eat healthily, they automatically start classifying foods into two (or maybe three) categories. These are usually healthy and unhealthy (and maybe a category for foods somewhere in the middle). Foods such as vegetables and fruit will get classified into the healthy category, whilst junk food will be assigned to the unhealthy category.

2. **Household items**

Generally speaking, certain household objects are classified when it comes to what room they belong in within a house. For example, a toilet brush, bath towels, and body wash are all usually classified by what they are used for and kept in the bathroom. However, cooking utensils, oven mitts, and food items will be kept in the kitchen, as they have different uses compared to bathroom items.

3. **Data Classification**

Data classification involves analyzing data and organizing it into categories based on certain characteristics, such as metadata, file type, contents, and more. Data classification helps companies to understand their data better, answer questions about their data, and more.

When should children learn about classification?

Students begin learning about classification before they start learning about numbers, as classifying and sorting activities do not need to involve numbers. For example, students can classify objects according to their colors, sizes, or shape. When a student is aged between three and four years old they should start to learn how to classify things based on one simple characteristic.

As students progress through their education, they will start to learn how to classify things according to more than one characteristic and will be able to classify things into multiple different categories, rather than just two.

Why is Classification in Math important for preschoolers?

Learning about classification in math is important for preschoolers because it teaches them a range of thinking skills and lays the foundations for future problem-solving activities. Classification in math teaches students how to understand certain things about the world around them and helps to prepare them for their math education in kindergarten and beyond. Classifying is a pre-numbers skill that is crucial to your students' development, as students learn about numerical concepts through classification and sorting, as well as develop the ability to group numbers and sets.

Five skills students develop while learning about classification

1. Classifying skills
2. Logical thinking skills
3. Counting skills
4. Problem-solving skills
5. Identification and comparison skills

Classification vs. Sorting

Classifying and sorting are often taught together, but whilst they are very similar to one another, they still differ slightly, and you should be aware of this before starting to teach your students. Read on to find out how they differ, and the importance of teaching your students how to sort objects as well as classify them.

What's the difference between classifying and sorting in math?

When students classify items, they are organizing them into predetermined categories. This means that generally, students will be looking for a specific characteristic in an item before placing it into the correct category. For example, if you're classifying 2D and

3D shapes, students will be actively looking for shapes that have 2D characteristics so that they know they can classify them into their 2D shape category.

When students are sorting items, they are simply identifying similarities and differences between a group of items and placing them into corresponding piles according to these similarities and differences. For example, if students had a pile of 20 colored items, they could sort through these items one by one and organize them into groups according to their colors, so red items would go in one pile, blue would go in another pile, and so on.

The importance of teaching preschoolers about sorting

Sorting is an important pre-number cognitive skill that students develop throughout preschool. It teaches students how to identify simple similarities and differences between a group of objects, such as their color or shape, even if they cannot verbalize what these similarities and differences are.

Five classifying and sorting activities to do with your class

1. Classify and sort 2D and 3D shapes

What you'll need:

- Plastic 2D shapes (circles, triangles, squares, and so on)
- Plastic 3D shapes (spheres, prisms, cubes, and so on)

What to do:

Place all of your plastic shapes into a pile and explain to your students that they will be classifying the shapes into two categories, 2D shapes, and 3D shapes. Discuss the characteristics of each, such as 2D shapes only having two dimensions (length and width) and 3D shapes having three dimensions (length, width, and height), 2D shapes having fewer faces, sides, and corners than 3D shapes, and the fact that 2D shapes are flat whilst 3D shapes are not.

Next, get your students to start identifying these characteristics in the plastic shapes that are in the pile and classifying them into their correct categories. This activity will help your students to improve their shape identification, classification, and sorting skills, among others.

2. Classify and sort household objects

What you'll need:

- A variety of household objects (toothbrush, cutlery, plate, cooking utensils, and so on)

What to do:

Gather your household items and place them into one big pile. Explain to your class that they will be sorting through these items and deciding whether they belong in the "kitchen" pile or the "bathroom" pile. To decide which pile the object belongs in, students must identify the objects and their characteristics, before thinking about where they might find those objects in their own homes.

When students have decided which rooms the household items would be found in, they can place the items into that room's pile. This activity will help students to develop their reasoning and logic skills as well as their sorting and classification skills.

3. Color classification and sorting activities

What you'll need:

- An array of differently colored objects (red, blue, orange, green, pink, and so on)
- A piece of card stock for each color that you have included in your items

What to do:

Place all of your differently colored objects into a pile and explain to your students that they will be sorting and classifying these objects according to their colors. Spread out your different colored card stock, so students can place the colored objects onto the corresponding pieces of card stock.

Students must first identify the color of the object they have selected, and then place it on the piece of card stock that is the same color. This activity will improve your students' color identification skills, sorting skills, classification skills, and logic and reasoning skills, too.

4. Tidy up your classroom

What you'll need:

- A messy classroom!

What to do:

At the end of your lesson (or day, depending on how you want to complete this activity) when your classroom supplies are everywhere, explain to your students that they are going to be responsible for putting everything away into their correct places. This activity involves classifying items according to fixed criteria, as different classroom supplies will have specified places within your classroom.

For example, pencils, coloring pencils, rulers, erasers, and paper, can all be put away in the correct places, building blocks can be returned to the play area, books returned to the bookshelves in the reading corner, and so on. This activity might not be the most fun for your students, but it's a hands-on way of learning how items can be classified according to their characteristics and will improve your students' organizational skills, too!

5. Classify animals according to their animal groups

What you'll need:

- Plastic toy animals from each animal group (reptile, amphibian, fish, mammal, and bird)
- Five pieces of card stock
- Marker pen

What to do:

Before you begin this activity, write "reptile" onto one piece of your card stock, "amphibian" onto another, and "fish", "mammal", and "bird" onto your remaining pieces of card stock. Once you've done this, explain to your students that they will be classifying animals according to their animal groups and that each plastic toy animal must be classified and placed onto the piece of card stock that has that animal's group name written on it. For example, any toy birds must be placed onto the piece of card stock that has birds written on it, dogs, cats, and other mammals must be placed on the piece of card stock that has mammals on it, and so on. Discuss what different characteristics each animal has, and what characteristics make them belong to each group with your students as you complete this activity.

§11. Expressions with numbers and variables. The value of the numeric expression. Numerical equality and properties. Numerical inequalities and their properties. Univariate linear equation and its solution. Understanding inequality. Univariate linear inequality and its solution, displaying the solution on the coordinate axis.

Expressions with numbers and variables.

An algebraic expression comprises both numbers and variables together with at least one arithmetic operation.

Example

$4\cdot x-3$

A variable, as we learned in pre-algebra, is a letter that represents unspecified numbers. One may use a variable in the same manner as all other numerals:

Addition	$4+y$	4 plus y
Subtraction	$x-5$	x minus 5
	$8-a$	8 minus a
Division	$z/7$	z divided by 7
	$14/x$	14 divided by x
Multiplication	$9x$	9 times x

To evaluate an algebraic expression you have to substitute each variable with a number and perform the operations included.

Example

Evaluate the expression when $x=5$

$4\cdot x-3$

First we substitute x with 5

$4\cdot 5-3$

And then we calculate the answer

$20-3=17$

An expression that represents repeated multiplication of the same factor is called a power e.g.

$5\cdot5\cdot5=125$

A power can also be written as

$5^3=125$

Where 5 is called the base and 3 is called the exponent. The exponent corresponds to the number of times the base is used as a factor.

$5^3=5\cdot5\cdot5$

31	3 to the first power	3
4^2	4 to the second power or 4 squared	$4\cdot4$
5^3	5 to the third power or 5 cubed	$5\cdot5\cdot5$
2^6	2 to the sixth power	$2\cdot2\cdot2\cdot2\cdot2\cdot2$

Why do we have math if we can describe things in words?

Algebraic expressions are useful because they represent the value of an expression for all of the values a variable can take on. Sometimes in math, we describe an expression with a phrase. For example, the phrase

"two more than five"

can be written as the expression

$5+2$, plus, 2.

Similarly, when we describe an expression in words that includes a variable, we're describing an algebraic expression, an expression with a variable.

For example,

"three more than x"

can be written as the algebraic expression

x+ 3, plus, 3.

But why? Why use math if we can describe things in words? One of the many reasons is that math is more precise and easier to work with than words are. This is a question you should keep thinking about as we dig deeper into algebra.

Different words for addition, subtraction, multiplication, and division

Here is a table that summarizes common words for each operation:

Operation	**Words**	**Example algebraic expression**
Addition	Plus, sum, more than, increased by	x + 3 x, plus, 3
Subtraction	Subtracted, minus, difference, less than, decreased by	p – 6 p, minus, 6
Multiplication	Times, product	8k 8, k
Division	Divided, quotient	a \div 9 a÷9 a, divided by, 9

For example, the word product tells us to use multiplication. So, the phrase

"the product of eight and k"

can be written as

8k .

Let's take a look at a trickier example

Write an expression for "m decreased by seven".

Notice that the phrase "decreased by" tells us to use subtraction.

So, the expression is m – 7   m, minus, 7.

[Why isn't the answer 7 - m?]

Linear equations and inequalities

Many problems that arise in the applications of mathematics lead naturally to equations. Equations in which the unknown only occurs to the first power are called **linear equations**. Thus

$2x-4=6$and $5x^2-7=4x+12$ 2x−4=6 and 5x2−7=4x+12

are examples of linear equations, whereas

x2−5x+6=0 and   2xx−1=x−32x+5x   2−5x+6=0   and 2xx−1=x−32x+5

are not linear, but are examples of quadratic equations which will be dealt with later. Equations such as

$x^3-3x2+6x-4=0$ $x^3-3x2+6x-4=0$

are called **polynomial equations**, and equations such as

$e^x=x-3e^x=x-3$

are often referred to as **transcendental equations**. With some exceptions, such transcendental equations are generally not dealt with in secondary school mathematics.

It is important to distinguish clearly between identities and equations. An **identity** is a statement that is true for (almost) all values of the pronumeral. For example,

$(x+1)^2=x^2+2x+1$ $(x+1)^2=x^2+2x+1$

is an identity, since when any real number is substituted for xx a true statement results. An **equation** is generally only true for certain values of the pronumeral. Thus, 3x−2=10

$3x-2=10$ only yields a true statement when xx takes the value 4. This is called the **solution** of the equation.

Linear equations only have (at most) one solution. Can you write down a linear equation with no solution?

An **equation** is a statement indicating that two algebraic expressions are equal. A **linear equation with one variable**, x, is an equation that can be written in the standard form $ax+b=0$ where a and b are real numbers and $a\neq0$. For example

$3x-12=0$

A **solution** to a linear equation is any value that can replace the variable to produce a true statement. The variable in the near equation $3x-12=0$ is x and the solution is $x=4$. To verify this, substitute the value 4 in for xx and check that you obtain a true statement.

$3x-12=0$

$3(4)-12=0$

✓$3x-12=0$

$3(4)-12=0$

$12-12=0$

$0=0$ ✓

Alternatively, when an equation is equal to a constant, we may verify a solution by substituting the value in for the variable and showing that the result is equal to that constant. In this sense, we say that solutions "satisfy the equation."

Example

Is $a=2$ solution to $-10a+5=-25$?

Solution

Recall that when evaluating expressions, it is a good practice to first replace all variables with parentheses, and then substitute the appropriate values. By making use of parentheses, we avoid some common errors when working the order of operations.

$−10a+5=−25$

$-10(2) +5=−25$

$-20+5+-25$

$-15\neq25$✗

Answer:

No, a=2 does not satisfy the equation and is therefore not a solution.

Developing techniques for solving various algebraic equations is one of our main goals in algebra. This section reviews the basic techniques used for solving linear equations with one variable. We begin by defining **equivalent equations** as equations with the same solution set.

$3x−5=16$

$3x=21$

x=7⎩

⎫⎭⎧Equivalent equations

$3x−5=16$

$3x=21$

x=7} **Equivalent equations**

Here we can see that the three linear equations are equivalent because they share the same solution set, namely {7}. To obtain equivalent equations, use the following **properties of equality**. Given algebraic expressions A and B, where c is a nonzero number:

Addition property of equality:	If $A=BA=B$, then $A+c=B+c$
Subtraction property of equality:	If $A=BA=B$, then $A-c=B-c$
Multiplication property of equality:	If $A=BA=B$, then $cA=cB$
Division property of equality:	If $A=B$ then $A/c=B/c$

The various types of linear equations and the various strategies to solve them are dealt with at length in the module Linear equations (Years 7–8), and so we will only quickly revise some of these ideas here via two examples. The basic rule throughout is that whatever you do to one side of the equation you must also do to the other.

Example

$2x+14=3$

Solution

$$\left\{\begin{array}{l} 2x + 14 = 3 \\ 2x = 3 - 14 \\ 2x = -11 \\ x = -\frac{11}{2} \\ x = -5.5 \end{array}\right. \quad \textbf{Equivalent equations}$$

Linear inequalities

A linear inequality resembles in form an equation, but with the equal sign replaced by an inequality symbol. The solution of a linear inequality is generally a range of values, rather than one specific value. Such inequalities arise naturally in problems involving words such as 'at least' or 'at most'.

To solve an inequality we use the same procedures as we used when solving linear equations, with the modification that when an inequality is multiplied or divided by a negative number, the inequality is reversed.

Inequalities also arise when we examine the domain of certain functions. Hence it is important that students are familiar with these before studying functions.

Understanding inequality. Univariate linear inequality and its solution, displaying the solution on the coordinate axis.

Definition

An inequality is a mathematical relationship between two expressions and is represented using one of the following:

- ≤: "less than or equal to"
- <: "less than"
- ≠: "not equal to"
- >: "greater than"
- ≥: "greater than or equal to"

Inequalities involving "<", "≠" or ">" are referred to as "strict inequalities", while inequalities involving "≤" or "≥" are not.

If you "switch" the two sides of an inequality you must then reverse the direction of the inequality symbol. For example since it is true that $4 < 5$, it is also true that $5 > 4$.

An equation is a statement of equality between two expressions. An equation uses the equality symbol (=).

Like solutions to conditional equations, solutions to inequalities in one variable can be represented using a number line.

When considering locations along a number line, the inequality symbols can be interpreted as follows:

- ≤: "to the left of or coincidental with" or "to the left of including"
- <: "to the left of"
- ≠: "not coincidental with"
- >: "to the right of"
- ≥: "to the right of or coincidental with" or "to the right of including"

Strict inequalities are usually used when no variables are involved.

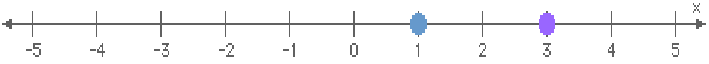
Inequalities involving a variable are sometimes referred to as "inequations".

At the present time, the term "inequality" refers to both inequations (with a variable) and simple inequalities without a variable.

Examples (Inequalities Without a Variable)

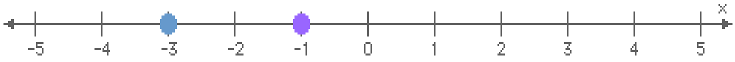
INEQUALITY: $1 < 3$ One is less than three

NUMBER LINE: $1 < 3$ One is to the left of three



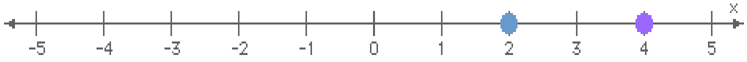
INEQUALITY: $-3 < -1$ Negative three is less than negative one

NUMBER LINE: $-3 < -1$ Negative three is to the left of negative one



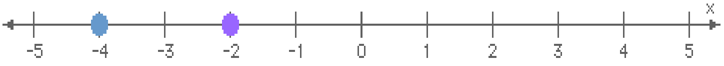
INEQUALITY: $4 > 2$ Four is greater than two

NUMBER LINE: $4 > 2$ Four is to the right of two



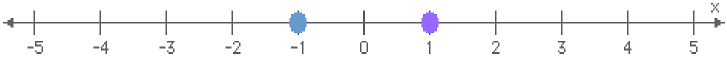
INEQUALITY: $-2 > -4$ Negative two is greater than negative four

NUMBER LINE: $-2 > -4$ Negative two is to the right of negative four



INEQUALITY: $-1 \neq 1$ Negative one is not equal to positive one

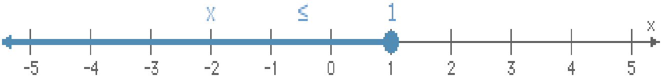
NUMBER LINE: $-1 \neq 1$ Negative one is not at the same location as positive one



Examples (Inequalities With a Variable)

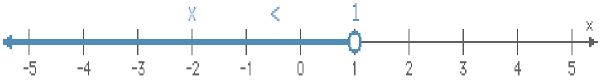
LEFT OF ONE INCLUDING ONE

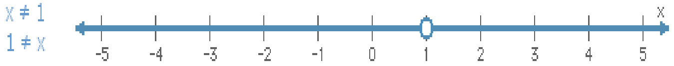
$x \leq 1$
$1 \geq x$

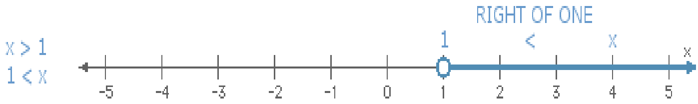


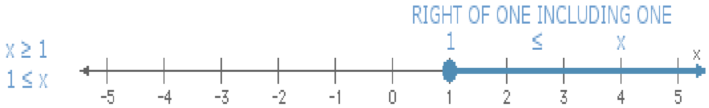
LEFT OF ONE

$x < 1$
$1 > x$





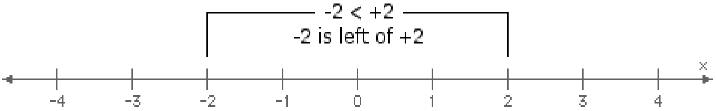




Demonstration

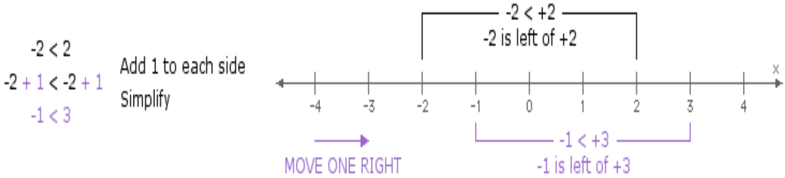
Multiplying (or dividing) both sides of an inequality by a negative number

-2 < 2: -2 is to the left of +2 on the number line (as shown below).

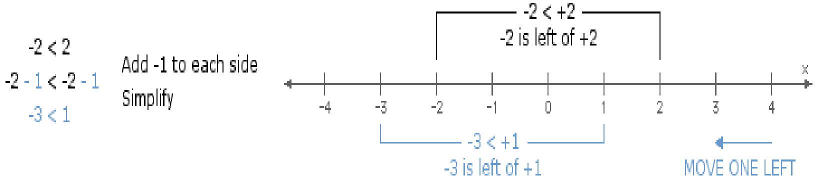


Adding or subtracting either a positive or negative number to each side of the inequality will result in a true statement.

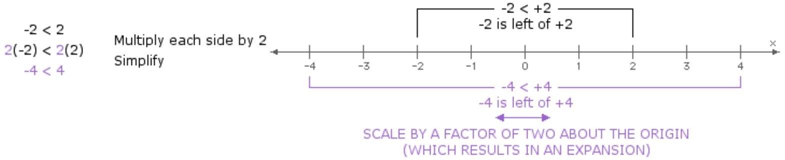
For example, if you add one to each side of the inequality (or equivalently subtract negative one from each side of the inequality), the following occurs:



If you add negative one to each side of the inequality (or equivalently subtract positive one from each side of the inequality), the following occurs:



If you multiply each side of the inequality by two, the following occurs:



IMPORTANT

If you multiply each side of the inequality by negative two, the following occurs:

$-2 < 2$

✗ $-2(-2) < -2(2)$ Multiply each side by -2

$4 < -4$ Simplify

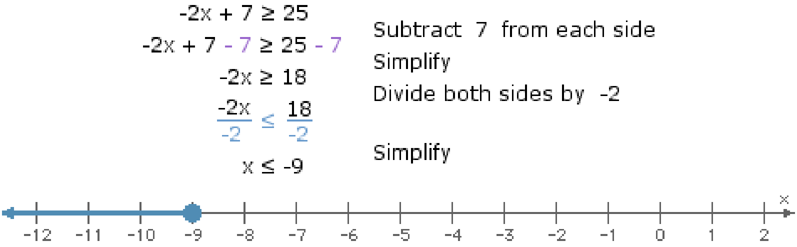
Note that the result of multiplying (or dividing) both sides of a (true) inequality by a negative number is an inequality which is false, unless you reverse the direction of the inequality.

$-2 < 2$

✓ $-2(-2) > -2(2)$ Multiply each side by -2

$4 > -4$ Simplify

It is important to keep this in mind when solving an inequality such as $-2x + 7 \geq 25$.



$-2x + 7 \geq 25$

$-2x + 7 - 7 \geq 25 - 7$ Subtract 7 from each side

$-2x \geq 18$ Simplify

$\frac{-2x}{-2} \leq \frac{18}{-2}$ Divide both sides by -2

$x \leq -9$ Simplify

§12. Basic geometric concepts. Plane and space figures, descriptions, properties

You use geometric terms in everyday language, often without thinking about it. For example, any time you say “walk along this line” or “watch out, this road quickly angles to the left” you are using geometric terms to make sense of the environment around you. You use these terms flexibly, and people generally know what you are talking about.

In the world of mathematics, each of these geometric terms has a specific definition. It is important to know these definitions—as well as how different figures are constructed—to become familiar with the language of geometry. Let’s start with a basic geometric figure: the plane.

Figures on a Plane

A **plane** is a flat surface that continues forever (or, in mathematical terms, infinitely) in every direction. It has two dimensions: length and width.

You can visualize a plane by placing a piece of paper on a table. Now imagine that the piece of paper stays perfectly flat and extends as far as you can see in two directions, left-to-right and frontto-back. This gigantic piece of paper gives you a sense of what a geometric plane is like: it continues infinitely in two directions.

(Unlike the piece of paper example, though, a geometric plane has no height.)

A plane can contain a number of geometric figures. The most basic geometric idea is a **point**, which has no dimensions. A point is simply a location on the plane. It is represented by a dot. Three points that don't lie in a straight line will determine a plane.

The image below shows four points, labeled A, B, C, and D.

Figure : A set of points

Two points on a plane determine a line. A line is a one-dimensional figure that is made up of an infinite number of individual points placed side by side. In geometry, all lines are assumed to be straight; if they bend they are called a curve. A line continues infinitely in two directions.

Below is line AB or, in geometric notation, AB←→AB↔. The arrows indicate that the line keeps going forever in the two directions. This line could also be called line BA. While the order of the points does not matter for a line, it is customary to name the two points in alphabetical order.

The image below shows the points A and B and the line AB←→AB↔.

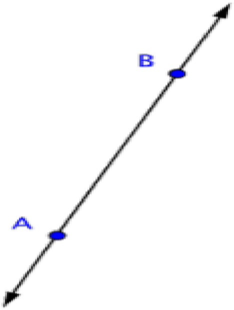
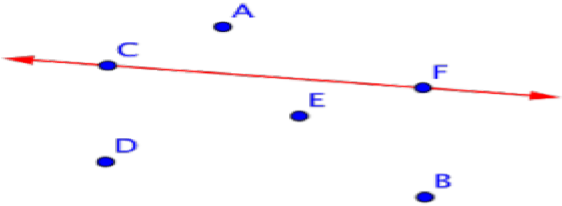


Figure : Line AB

Example

Name the line shown in red.



Solution

The red line goes through the points C and F, so the line is CF←→CF↔.

Answer: CF←→CF↔

There are two more figures to consider. The section between any two points on a line is called a line segment.

A line segment can be very long, very short, or somewhere in between. The difference between a line and a **line segment** is that the line segment has two endpoints and a line goes on forever. A line segment is denoted by its two endpoints, as in CD¯¯¯¯¯¯¯¯CD¯.

Figure : Line segment CD

A **ray** has one endpoint and goes on forever in one direction. Mathematicians name a ray with notation like EF−→−EF→, where point E is the endpoint and F is a point on the ray. When naming a ray, we always say the endpoint first. Note that FE−→−FE→ would have the endpoint at F, and continue through E, which is a different ray than EF−→−EF→, which would have an endpoint at E, and continue through F.

The term "ray" may be familiar because it is a common word in English. "Ray" is often used when talking about light. While a ray of light resembles the geometric term "ray," it does not go on forever, and it has some width. A geometric ray has no width; only length.

Below is an image of ray EF or EF−→−EF→. Notice that the endpoint is E.

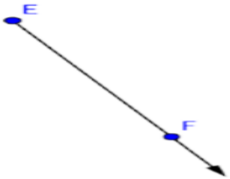
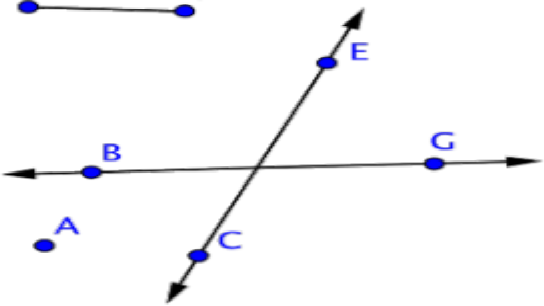


Figure : Ray EF

Example

Identify each line and line segment in the picture below.



Solution

Two points define a line, and a line is denoted with arrows. There are two lines in this picture:

CE←→CE↔ and BG←→BG↔.

A line segment is a section between two points. DF¯¯¯¯¯¯DF¯ is a line segment. But there are also two more line segments on the lines themselves: CE¯¯¯¯¯¯CE¯ and BG¯¯¯¯¯¯BG¯.

Answer: Lines:

CE¯¯¯¯¯¯CE¯,

BG¯¯¯¯¯¯BG¯

Line segments:

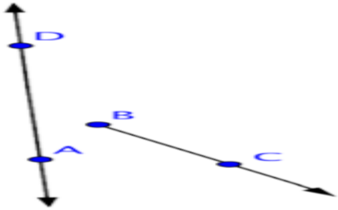
DF¯¯¯¯¯¯DF¯,

CE¯¯¯¯¯¯CE¯,

BG¯¯¯¯¯¯BG¯.

Example

Identify each point and ray in the picture below.



Solution

There are four points: A, B, C, and D. There are also three rays, though only one may be obvious.

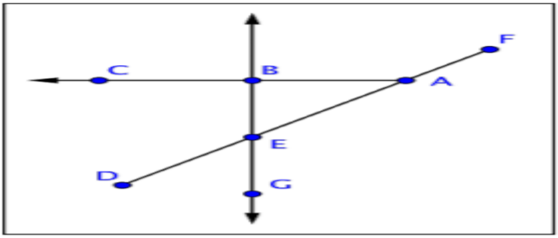
Ray BC−→−BC→ begins at point B and goes through C. Two more rays exist on line AD←→AD↔: they are DA−→−DA→ and AD−→−AD→.

Answer: Points: A, B, C, D

Rays: BC−→−BC→, AD−→−AD→, DA−→−DA→

Try Ir Now 1

Which of the following is not represented in the image below?



A) BG

B) BA

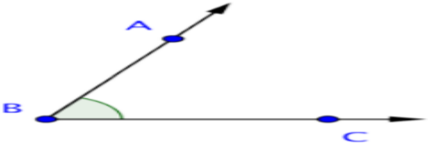
C) DF¯¯¯¯¯¯¯¯DF¯

D) AC

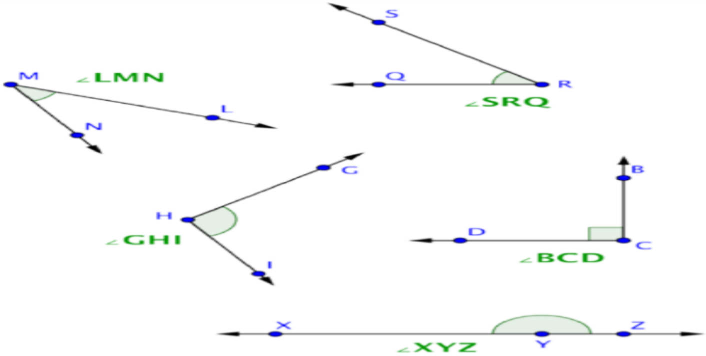
Angles

Lines, line segments, points, and rays are the building blocks of other figures. For example, two rays with a common endpoint make up an **angle**. The common endpoint of the angle is called the **vertex**.

The angle ABC is shown below. This angle can also be called ∠ABC, ∠CBA or simply ∠B. When you are naming angles, be careful to include the vertex (here, point B) as the middle letter.



The image below shows a few angles on a plane. Notice that the label of each angle is written "point-vertex-point," and the geometric notation is in the form ∠ABC.



Sometimes angles are very narrow; sometimes they are very wide. When people talk about the "size" of an angle, they are referring to the arc between the two rays. The length of the rays has nothing to do with the size of the angle itself. Drawings of angles will often include an arc (as shown above) to help the reader identify the correct 'side' of the angle.

Think about an analog clock face. The minute and hour hands are both fixed at a point in the middle of the clock. As time passes, the hands rotate around the fixed point, making larger and smaller angles as they go. The length of the hands does not impact the angle that is made by the hands.

An angle is measured in degrees, represented by the symbol º. A circle is defined as having 360º. (In skateboarding and basketball, “doing a 360” refers to jumping and doing one complete body rotation.

A **right angle** is any degree that measures exactly 90º. This represents exactly one-quarter of the way around a circle. Rectangles contain exactly four right angles. A corner mark is often used to denote a right angle, as shown in right angle DCB below.

Angles that are between 0º and 90º (smaller than right angles) are called **acute angles**. Angles that are between 90º and 180º (larger than right angles and less than 180º) are called **obtuse angles**. And an angle that measures exactly 180º is called a **straight angle** because it forms a straight line.

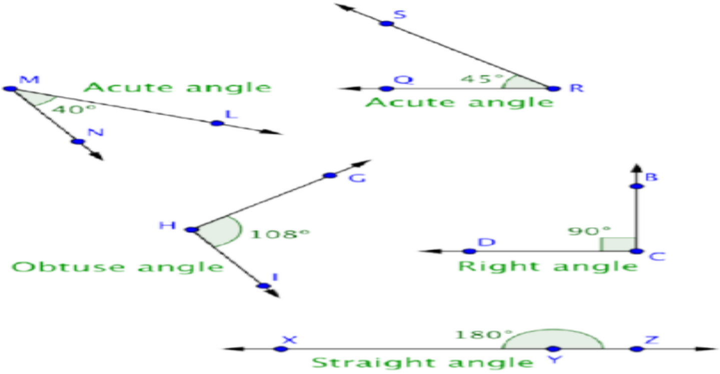
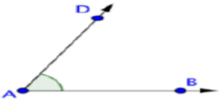
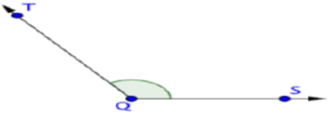


Figure .Examples of Angles

Example

Label each angle below as acute, right, or obtuse.





Solution

You can start by identifying any right angles.

∠GFI is a right angle, as indicated by the corner mark at vertex F.

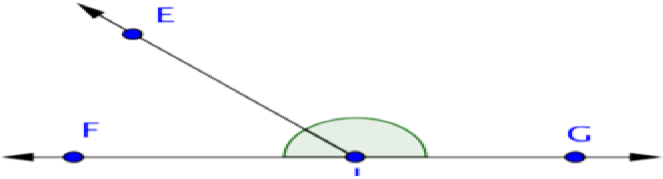
Acute angles will be smaller than ∠GFI (or less than 90º). This means that ∠DAB and ∠MLN are acute.

∠TQS is larger than ∠GFI, so it is an obtuse angle.

Answer: ∠DAB and ∠MLN are acute angles. ∠GFI is a right angle. ∠TQS is an obtuse angle.

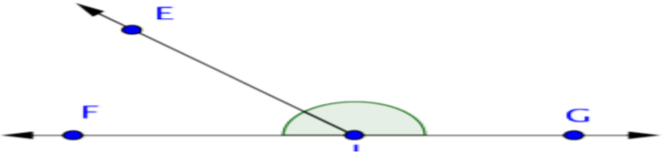
Example

Identify each point, ray, and angle in the picture below.

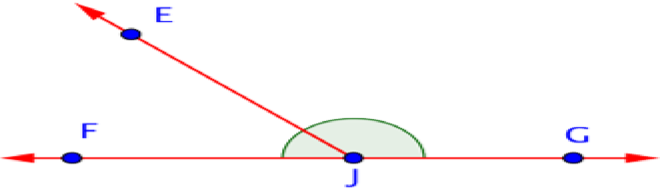


Solution

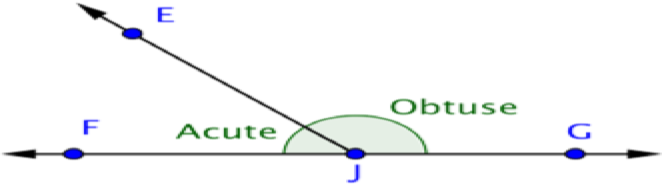
Begin by identifying each point in the figure. There are 4: E, F, G, and J.



Now find rays. A ray begins at one point, and then continues through another point towards infinity (indicated by an arrow). Three rays start at point J: JE−→JE→, JF−→JF→, and JG−→JG→. But also notice that a ray could start at point F and go through J and G, and another could start at point G and go through J and F. These rays can be represented by GF−→−GF→ and FG−→−FG→.



Finally, look for angles. ∠EJG is obtuse, ∠EJF is acute, and ∠FJG is straight. (Don’t forget those straight angles!)



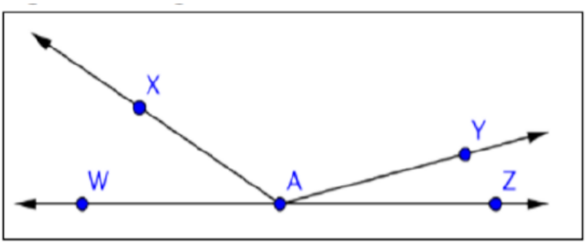
Answer: Points: E, F, G, J

Rays: JE−→JE→, JG−→JG→, JF−→JF→, GF−→−GF→, FG−→−FG→

Angles: ∠EJG, ∠EJF, ∠FJG

Try It Now 2

Identify the acute angles in the given image:



Measuring Angles with a Protractor

Learning how to measure angles can help you become more comfortable identifying the difference between angle measurements. For instance, how is a 135° angle different from a 45° angle?

Measuring angles requires a protractor, which is a semi-circular tool containing 180 individual hash marks. Each hash mark represents 1°. (Think of it like this: a circle is 360°, so a semi-circle is 180°.) To use the protractor, do the following three steps:

Step 1. Line up the vertex of the angle with the dot in the middle of the flat side (bottom) of the protractor,

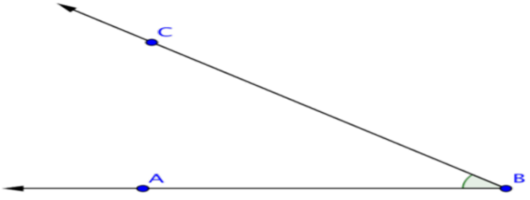
Step 2. Align one side of the angle with the line on the protractor that is at the zero-degree mark, and

Step 3. Look at the curved section of the protractor to read the measurement.

The example below shows you how to use a protractor to measure the size of an angle.

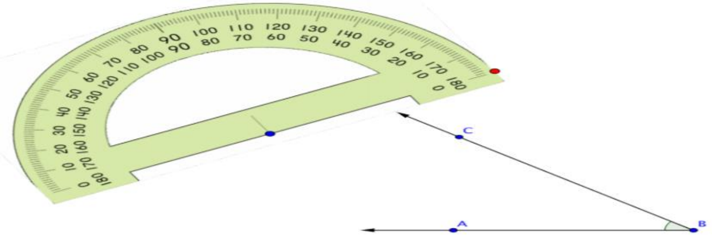
Example

Use a protractor to measure the angle shown below.

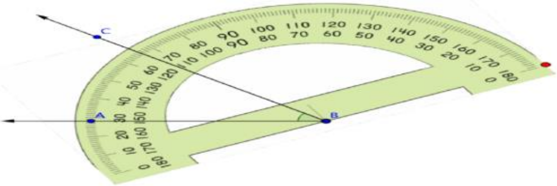


Solution

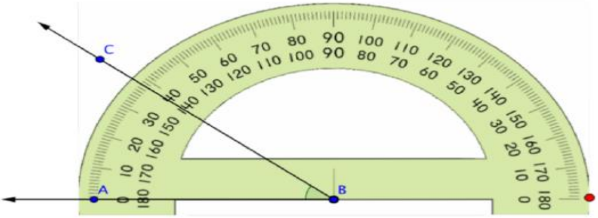
Use a protractor to measure the angle.



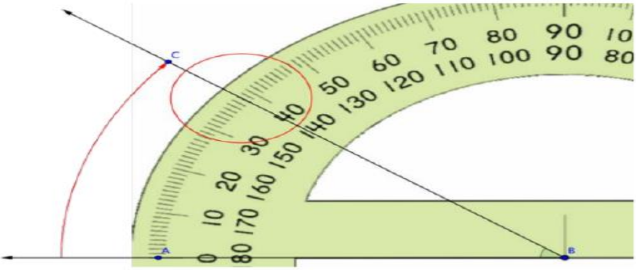
Align the blue dot on the protractor with the vertex of the angle you want to measure.



Rotate the protractor around the vertex of the angle until the side of the angle is aligned with the 0 degree mark of the protractor.



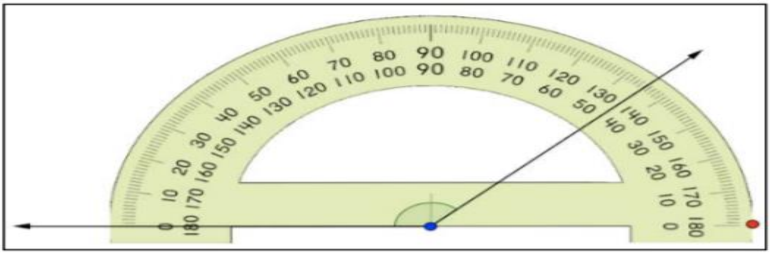
Read the measurement, in degrees, of the angle. Begin with the side of the angle that is aligned with the 0º mark of the protractor and count up from 0º. This angle measures 38º.



Answer: The angle measures 38º.

Try It Now 3

What is the measurement of the angle shown below?



Summary

Geometry begins with simple concepts like points, lines, segments, rays, etc. and expands with angles. As we can see from this section, there are multiple types of angles and several ways to measure them. The most accurate way of measuring an angle is using a protractor. When we put angles together, we obtain geometric shapes and solids, which we discuss in future sections. Next, we discuss lines, and using properties to obtain measures of angles.

Try It Now Answers

1. BA; this image does not show any ray that begins at point B and goes through point A.
2. ∠WAX and ∠YAZ x; both ∠WAX and ∠YAZ are acute angles.

3. 135º; this protractor is aligned correctly, and the correct measurement is 135º.

Properties of Angles

1. Identify parallel and perpendicular lines.
2. Find measures of angles.
3. Identify complementary and supplementary angles.

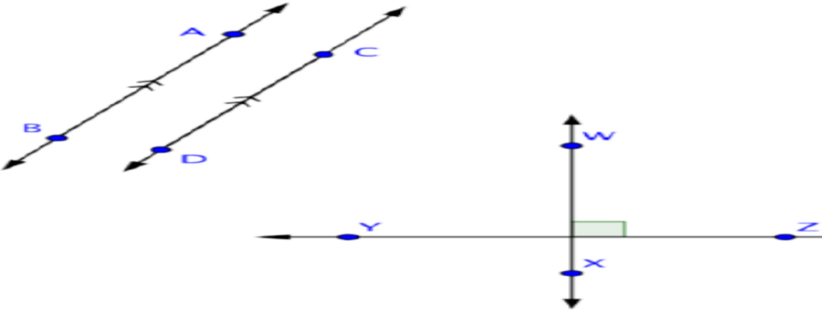
Imagine two separate and distinct lines on a plane. There are two possibilities for these lines: they will either intersect at one point, or they will never intersect. When two lines intersect, four angles are formed. Understanding how these angles relate to each other can help you figure out how to measure them, even if you only have information about the size of one angle.

Parallel and Perpendicular

Parallel lines are two or more lines that never intersect. Likewise, parallel line segments are two line segments that never intersect even if the line segments were turned into lines that continued forever. Examples of parallel line segments are all around you, in the two sides of this page and in the shelves of a bookcase. When you see lines or structures that seem to run in the same direction, never cross one another, and are always the same distance apart, there’s a good chance that they are parallel.

Perpendicular lines are two lines that intersect at a 90º (right) angle. And perpendicular line segments also intersect at a 90º (right) angle. You can see examples of perpendicular lines everywhere as well—on graph paper, in the crossing pattern of roads at an intersection, to the colored lines of a plaid shirt. In our daily lives, you may be happy to call two lines perpendicular if they merely seem to be at right angles to one another. When studying geometry, however, you need to make sure that two lines intersect at a 90º angle before declaring them to be perpendicular.

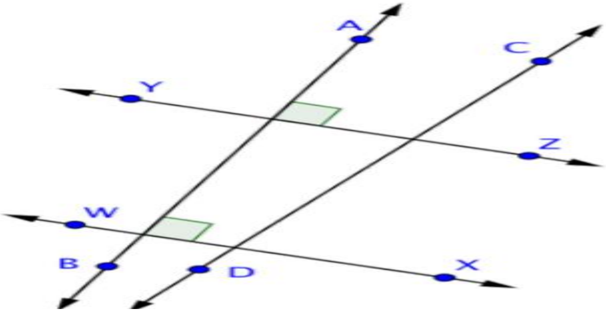
The image below shows some parallel and perpendicular lines. The geometric symbol for parallel is ||, so you can show that AB || CD. Parallel lines are also often indicated by the marking >> on each line (or just a single > on each line). Perpendicular lines are indicated by the symbol ⊥, so you can write WX←→−⊥YZ←→WX↔⊥YZ↔.



If two lines are parallel, then any line that is perpendicular to one line will also be perpendicular to the other line. Similarly, if two lines are both perpendicular to the same line, then those two lines are parallel to each other. Let's take a look at one example and identify some of these types of lines.

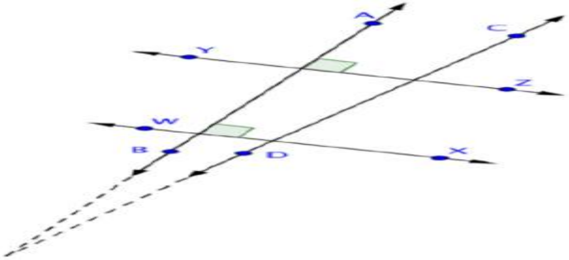
Example

Identify a set of parallel lines and a set of perpendicular lines in the image below.

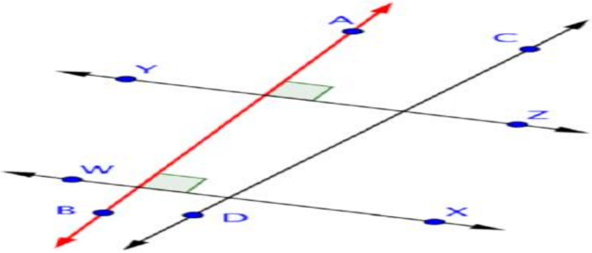


Solution

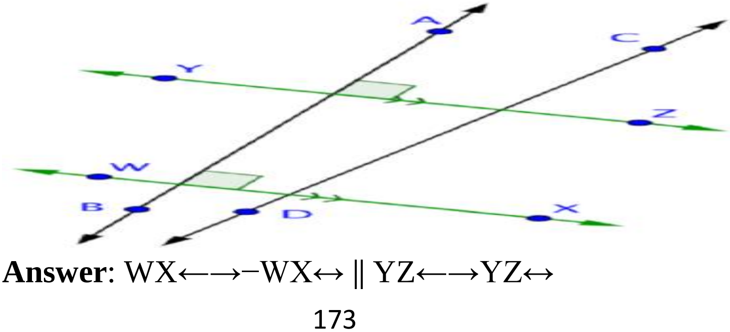
Parallel lines never meet, and perpendicular lines intersect at a right angle. AB←—→AB↔ and CD←—→CD↔ do not intersect in this image, but if you imagine extending both lines, they will intersect soon. So, they are neither parallel nor perpendicular.



AB←—→AB↔ is perpendicular to both WX←—→−WX↔ and YZ←—→YZ↔, as indicated by the right-angle marks at the intersection of those lines.



Since AB←—→AB↔ is perpendicular to both lines, then WX←—→−WX↔ and YZ←—→YZ↔ are parallel.

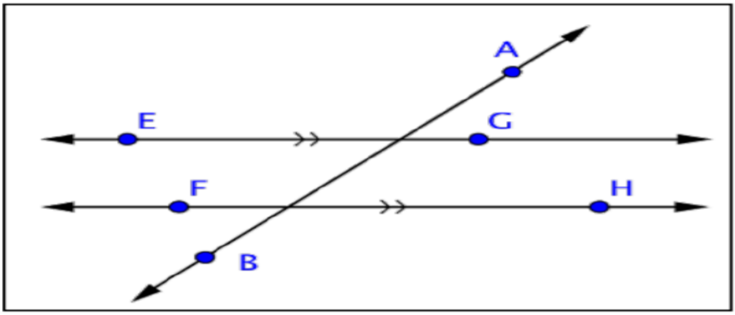


Answer: WX←—→−WX↔ ∥ YZ←—→YZ↔

AB←→AB↔ ⊥ WX←→−WX↔, AB←→AB↔ ⊥ YZ←→YZ↔

Try It Now 1

Which statement most accurately represents the image below?



A) EF || GH

B) AB ⊥ EG

C) FH || EG

D) AB || FH

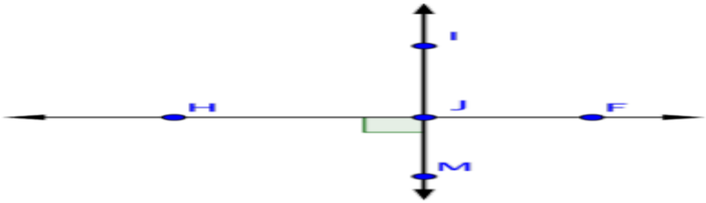
Finding Angle Measurements

Understanding how parallel and perpendicular lines relate can help you figure out the measurements of some unknown angles. To start, all you need to remember is that perpendicular lines intersect at a 90º angle and that a straight angle measures 180º.

The measure of an angle such as ∠A is written as m∠A. Look at the example below. How can you find the measurements of the unmarked angles?

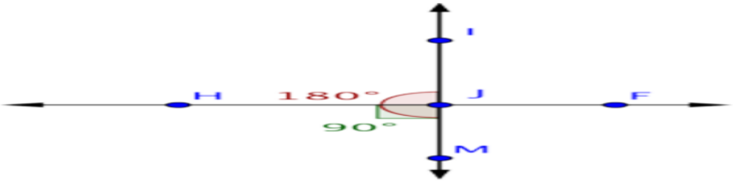
Example

Find the measurement of ∠IJF.



Solution

Only one angle, ∠HJM, is marked in the image. Notice that it is a right angle, so it measures 90º. ∠HJM is formed by the intersection of lines IM←—→IM↔ and HF←—→HF↔. Since IM←—→IM↔ is a line, ∠IJM is a straight angle measuring 180º.

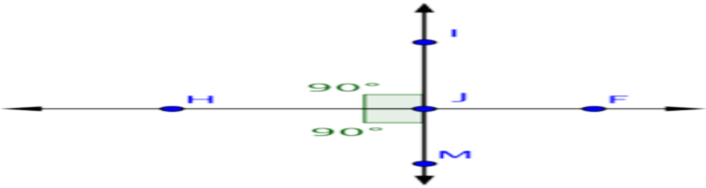


You can use this information to find the measurement of ∠HJI :

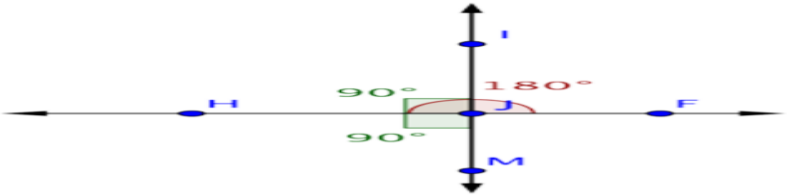
m∠HJM + m∠HJI = m∠IJM

90º + m∠HJI = 180º

m∠HJI = 90º



Now use the same logic to find the measurement of ∠IJF. ∠IJF is formed by the intersection of lines IM←—→IM↔ and HF←—→HF↔. Since HF←—→HF↔ is a line, ∠HJF will be a straight angle measuring 180º.

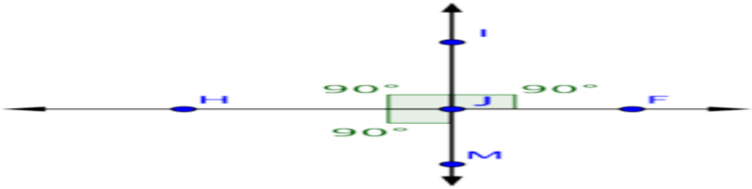


You know that ∠HJI measures 90º. Use this information to find the measurement of ∠IJF:

m∠HJM + m∠IJF = m∠HJF

90º + m∠IJF = 180º

m∠IJF = 90º



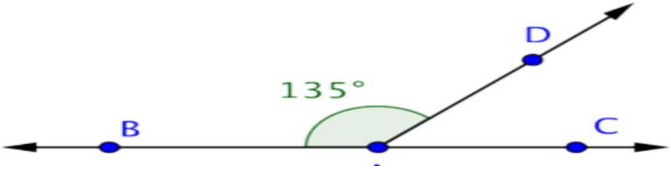
Answer: m∠IJF = 90º

In this example, you may have noticed that angles ∠HJI, ∠IJF, and ∠HJM are all right angles. (If you were asked to find the measurement of ∠FJM, you would find that angle to be 90º, too.) This is what happens when two lines are perpendicular—the four angles created by the intersection are all right angles.

Not all intersections happen at right angles, though. In the example below, notice how you can use the same technique as shown above (using straight angles) to find the measurement of a missing angle.

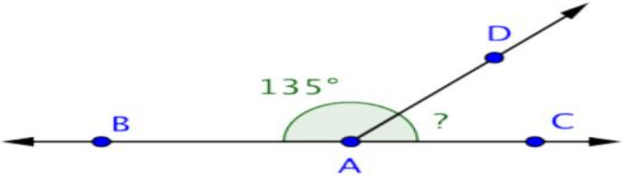
Example

Find the measurement of ∠DAC.

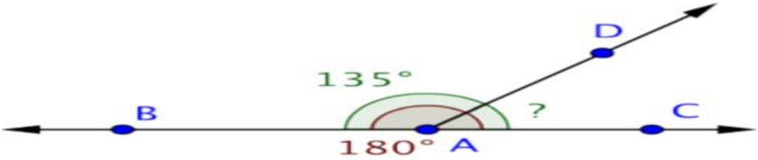


Solution

This image shows the line BC←—→BC↔ and the ray AD−→−AD→ intersecting at point A. The measurement of ∠BAD is 135º. You can use straight angles to find the measurement of ∠DAC.



∠BAC is a straight angle, so it measures 180º.

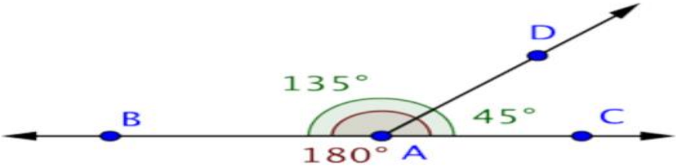


Use this information to find the measurement of ∠DAC.

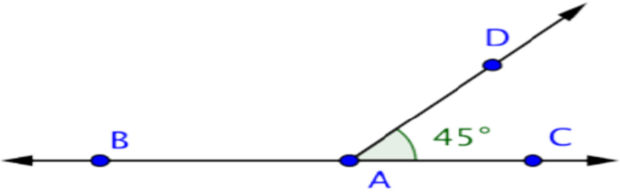
m∠BAD + m∠DAC = m∠BAC

135º + m∠DAC = 180º

m∠DAC = 45º

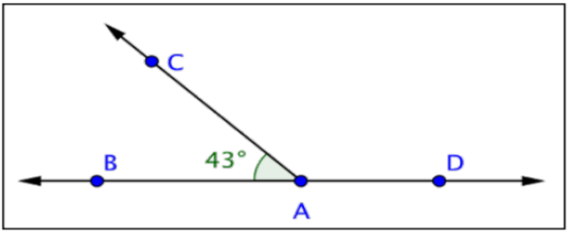


Answer: m∠DAC = 45º



Try It Now 2

Find the measurement of ∠CAD.



Supplementary and Complementary

In the example above, m∠BAC and m∠DAC add up to 180º. Two angles whose measures add up to 180º are called **supplementary angles**. There's also a term for two angles whose measurements add up to 90º, they are called **complementary angles**.

One way to remember the difference between the two terms is that "corner" and "complementary" each begin with c (a 90º angle looks like a corner), while straight and "supplementary" each begin with s (a straight angle measures 180º).

If you can identify supplementary or complementary angles within a problem, finding missing angle measurements is often simply a matter of adding or subtracting.

Example

Two angles are supplementary. If one of the angles measures 48º, what is the measurement of the other angle?

Solution

Two supplementary angles make up a straight angle, so the measurements of the two angles will be 180º.

$m\angle A + m\angle B = 180^{\circ}$

You know the measurement of one angle. To find the measurement of the second angle, subtract 48º from 180º.

$48^{\circ} + m\angle B = 180^{\circ}$

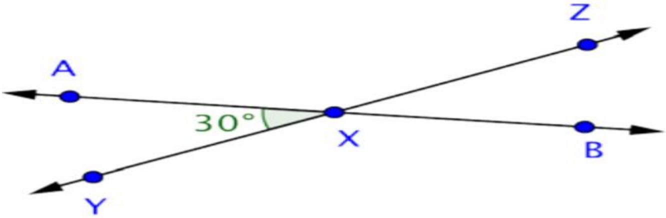
$m\angle B = 180^{\circ} - 48^{\circ}$

$m\angle B = 132^{\circ}$

Answer: The measurement of the other angle is 132º

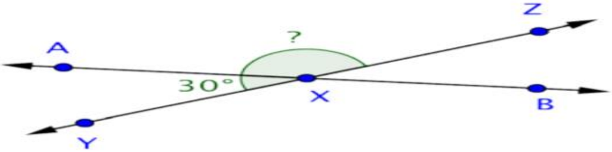
Example

Find the measurement of ∠AXZ.



Solution

This image shows two intersecting lines, AB←→AB↔ and YZ←→YZ↔. They intersect at point X, forming four angles. Angles ∠AXY and ∠AXZ are supplementary because together they make up the straight angle ∠YXZ.

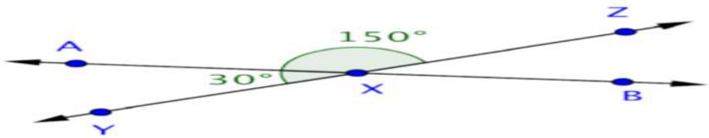


Use this information to find the measurement of ∠AXZ.

$m\angle AXY + m\angle AXZ = m\angle YXZ$

$30^{\circ} + m\angle AXZ = 180^{\circ}$

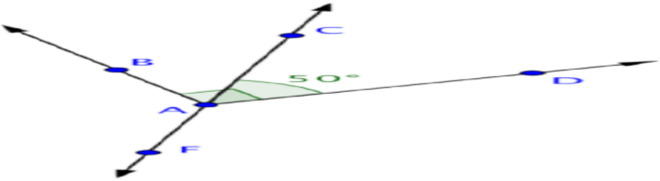
$m\angle AXZ = 150^{\circ}$



Answer: m∠AXZ = 150º

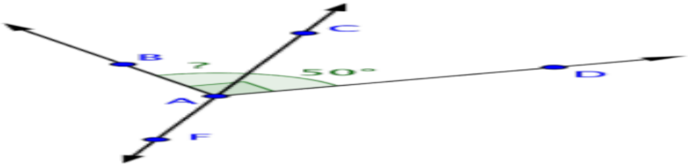
Example

Find the measurement of ∠BAC.



Solution

This image shows the line CF⟵⟶CF↔ and the rays AB⟵⟶AB↔ and AD⟵⟶AD↔, all intersecting at point A. Angle ∠BAD is a right angle. Angles ∠BAC and ∠CAD are complementary because together they create ∠BAD.

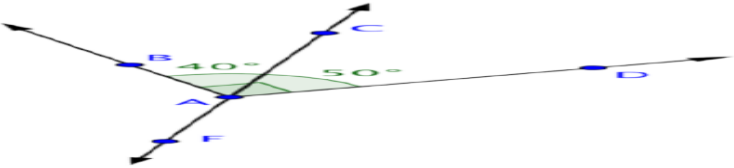


Use this information to find the measurement of ∠BAC .

m∠BAC + m∠CAD = m∠BAD

m∠BAC + 50º = 90º

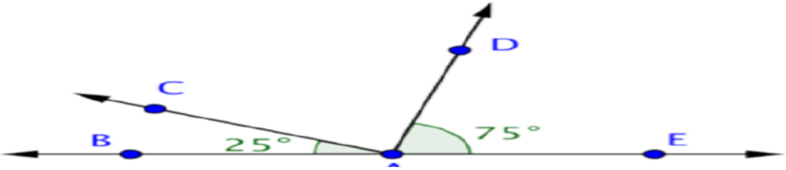
m∠BAC = 40º



Answer: m∠BAC = 40º

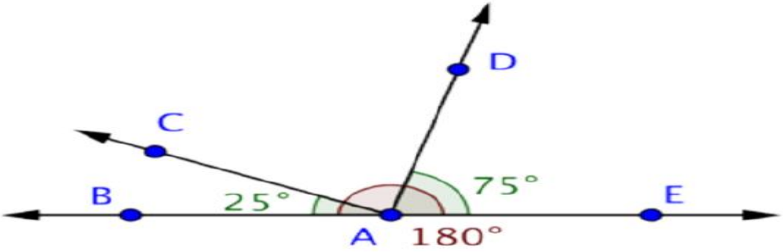
Example

Find the measurement of ∠CAD.



Solution

You know the measurements of two angles here: ∠CAB and ∠DAE. You also know that m∠BAE = 180º.



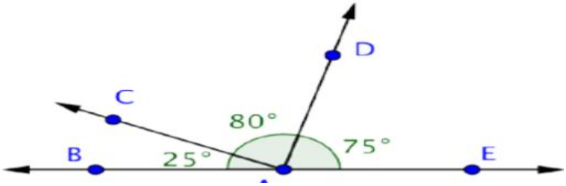
Use this information to find the measurement of ∠CAD.

m∠BAC + m∠CAD + m∠DAE = m∠BAE

25º + m∠CAD + 75º = 180º

m∠CAD + 100º = 180º

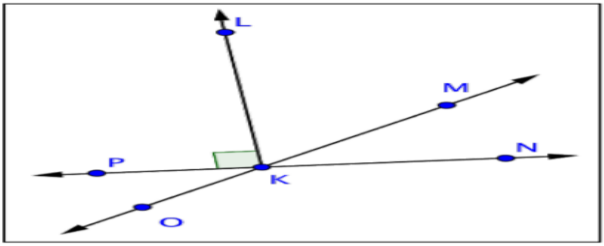
m∠CAD = 80º



Answer: m∠CAD = 80º

Try It Now 3

Which pair of angles is complementary?



A) ∠PKO and ∠MKN

B) ∠PKO and ∠PKM

C) ∠LKP and ∠LKN

D) ∠LKM and ∠MKN

Parallel lines do not intersect, while perpendicular lines cross at a 90° angle. Two angles whose measurements add up to 180° are said to be supplementary, and two angles whose measurements add up to 90° are said to be complementary. For most pairs of intersecting lines, all you need is the measurement of one angle to find the measurements of all other angles formed by the intersection.

Try It Now Answers

1. C) FH || EG; both EG and FH are marked with >> on each line, and those markings mean they are parallel.
2. 137°; ∠BAD is a straight angle measuring 180°. Since ∠BAC measures 43°, the measure of ∠CAD must be 180° – 43° = 137°.
3. D) ∠LKM and ∠MKN; the measurements of two complementary angles will add up to 90°. ∠LKP is a right angle, so ∠LKN must be a right angle as well. ∠LKM + ∠MKN = ∠LKN, so ∠LKM and ∠MKN are complementary.

Triangles

Learning Objectives

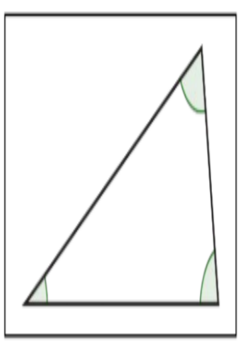
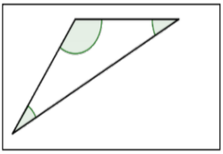
1. Identify equilateral, isosceles, scalene, acute, right, and obtuse triangles.
2. Identify whether triangles are similar, congruent, or neither.
3. Identify corresponding sides of congruent and similar triangles.
4. Find the missing measurements in a pair of similar triangles.
5. Solve application problems involving similar triangles

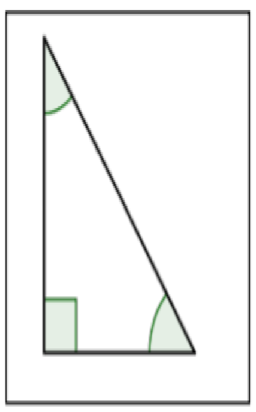
Geometric shapes, also called figures, are an important part of the study of geometry. The triangle is one of the basic shapes in geometry. It is the simplest shape within a classification of shapes called polygons. All triangles have three sides and three angles, but they come in many different shapes and sizes. Within the group of all triangles, the characteristics of a triangle's sides and angles are used to classify it even further. Triangles have some important characteristics, and understanding these characteristics allows you to apply the ideas in real-world problems.

Classifying and Naming Triangles

A polygon is a closed plane figure with three or more straight sides. Polygons each have a special name based on the number of sides they have. For example, the polygon with three sides is called a triangle because "tri" is a prefix that means "three." Its name also indicates that this polygon has three angles. The prefix "poly" means many.

The table below shows and describes three classifications of triangles. Notice how the types of angles in the triangle are used to classify the triangle.

Name of Triangle	**Picture of Triangle**	**Description**
Acute Triangle		A triangle with 3 acute angles (3 angles measuring between 0° and 90°).
Obtuse Triangle		A triangle with 1 obtuse angle (1 angle measuring between 90° and 180°).

Right Triangle		A triangle containing one right angle (1 angle that measures 90°). Note that the right angle is shown with a corner mark and does not need to be labeled 90°.

The sum of the measures of the three interior angles of a triangle is always 180°. This fact can be applied to find the measure of the third angle of a triangle, if you are given the other two. Consider the examples below.

Example

A triangle has two angles that measure 35° and 75°. Find the measure of the third angle.

Solution

The sum of the three interior angles of a triangle is 180°.

$35° + 75° + x = 180°$

Find the value of xx.

$110º + x= 180º$

$x = 180° – 110º$

$x = 70°$

Answer: The third angle of the triangle measures 70°.

Example

One of the angles in a right triangle measures 57º. Find the measurement of the third angle.

Solution

The sum of the three angles of a triangle is 180°. One of the angles has a measure of 90° as it is a right triangle.

$57° + 90° + x = 180°$

Simplify.

$147º + x = 180°$

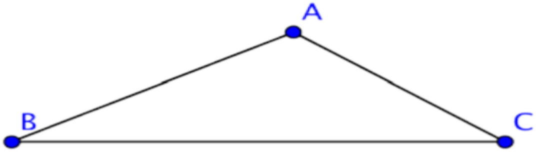
Find the value of x.

$x = 180º - 147º$

$x = 33º$

Answer: The third angle of the right triangle measures 33°.

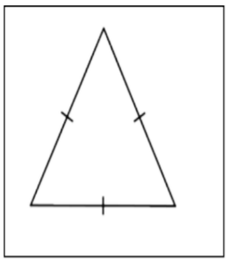
There is an established convention for naming triangles. The labels of the vertices of the triangle, which are generally capital letters, are used to name a triangle.

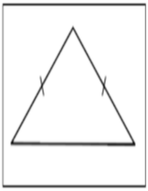
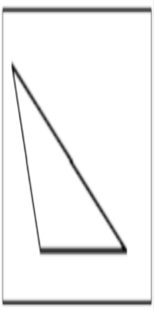


You can call this triangle ABC or ΔABC since A, B, and C are vertices of the triangle. When naming the triangle, you can begin with any vertex. Then keep the letters in order as you go around the polygon. The triangle above could be named in a variety of ways: ΔABC, or ΔCBA. The sides of the triangle are line segments AB, AC, and CB.

Just as triangles can be classified as acute, obtuse, or right based on their angles, they can also be classified by the length of their sides. Sides of equal length are called **congruent** sides. While we designate a segment joining points A and B by the notation AB¯¯¯¯¯¯¯¯AB¯, we designate the length of a segment joining points A and B by the notation AB without a segment bar over it. The length AB is a number, and the segment AB¯¯¯¯¯¯¯¯AB¯ is the collection of points that make up the segment.

Mathematicians show congruency by putting a hash mark symbol through the middle of sides of equal length. If the hash mark is the same on one or more sides, then those sides are congruent. If the sides have different hash marks, they are not congruent. The table below shows the classification of triangles by their side lengths.

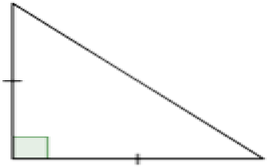
Name of Triangle	**Picture of Triangle**	**Description**
Equilateral Triangle		A triangle whose three sides have the same length. These sides of equal length are called congruent sides.

Isosceles Triangle		A triangle with exactly two congruent sides.
Scalene Triangle		A triangle in which all three sides are a different length.

To describe a triangle even more specifically, you can use information about both its sides and its angles. Consider this example.

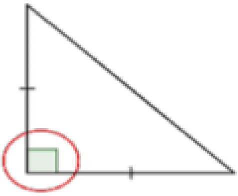
Example

Classify the triangle below.

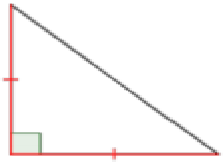


Solution

Notice what kind of angles the triangle has. Since one angle is a right angle, this is a right triangle.



Notice the lengths of the sides. Are there congruence marks or other labels?

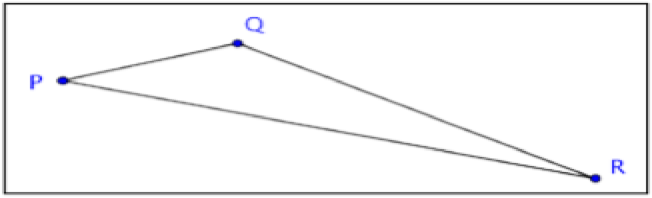


The congruence marks tell us there are two sides of equal length. So, this is an isosceles triangle.

Answer: This is an isosceles right triangle

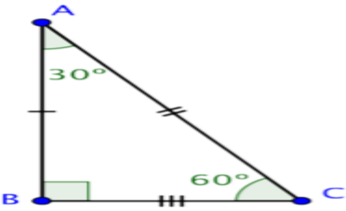
Try It Now 1

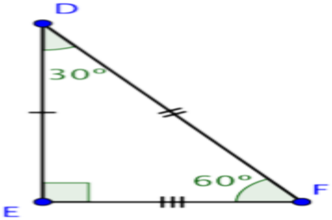
Classify the given triangle.



Identifying Congruent and Similar Triangles

Two triangles are congruent if they are exactly the same size and shape. In congruent triangles, the measures of **corresponding angles** and the lengths of **corresponding sides** are equal. Consider the two triangles shown below:





Since both ∠B and ∠E are right angles, these triangles are right triangles. Let’s call these two triangles ΔABC and ΔDEF. These triangles are congruent if every pair of corresponding sides has equal lengths and every pair of corresponding angles has the same measure.

The corresponding sides are opposite the corresponding angles.

↔ means “corresponds to”

∠B ↔ ∠E

∠A ↔ ∠D

∠C ↔ ∠F

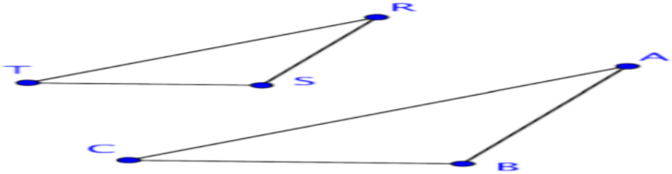
AB‾‾‾‾‾‾‾‾AB‾ ↔ DE‾‾‾‾‾‾‾‾DE‾

AC‾‾‾‾‾‾‾‾AC‾ ↔ DF‾‾‾‾‾‾‾‾DF‾

BC‾‾‾‾‾‾‾‾BC‾ ↔ EF‾‾‾‾‾‾‾‾EF‾

ΔABC and ΔDEF are congruent triangles as the corresponding sides and corresponding angles are equal.

Let’s take a look at another pair of triangles. Below are the triangles ΔABC and ΔRST.



These two triangles are surely not congruent because ΔRST is clearly smaller in size than ΔABC. But, even though they are not the same size, they do resemble one another. They are the same shape. The corresponding angles of these triangles look like they might have the same exact measurement, and if they did they would be congruent angles and we would call the triangles similar triangles.

Congruent angles are marked with hash marks, just as congruent sides are.

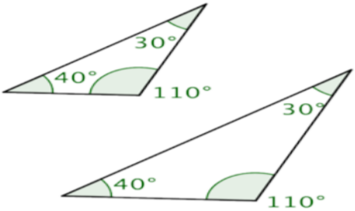


Figure : Image showing angle measurements of both triangles.

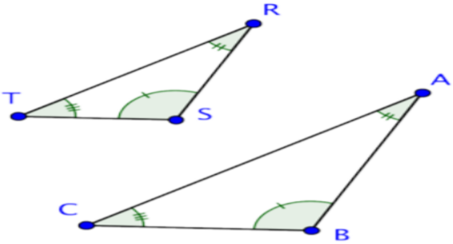
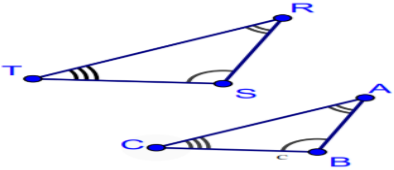


Figure : Image showing triangles ABC and RST using hash marks to show angle congruency.

We can also show congruent angles by using multiple bands within the angle, rather than multiple hash marks on one band. Below is an image using multiple bands within the angle.



Figure

Image showing triangles ABC and RST using bands to show angle congruency.

If the corresponding angles of two triangles have the same measurements they are called **similar triangles**. This name makes sense because they have the same shape, but not necessarily the same size. When a pair of triangles is similar, the corresponding sides are proportional to one another. That means that there is a consistent scale factor that can be used to compare the corresponding sides. In the previous example, the side lengths of the larger triangle are all 1.4 times the length of the smaller. So, similar triangles are proportional to one another.

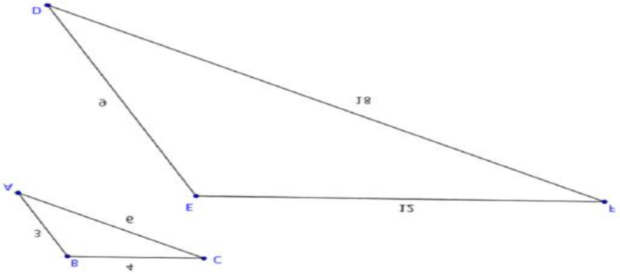
Just because two triangles look similar does not mean they are similar triangles in the mathematical sense of the word. Checking that the corresponding angles have equal measure is one way of being sure the triangles are similar.

Corresponding Sides of Similar Triangles

There is another method for determining similarity of triangles that involves comparing the ratios of the lengths of the corresponding sides.

If the ratios of the pairs of corresponding sides are equal, the triangles are similar.

Consider the two triangles below.



ΔABC is not congruent to ΔDEF because the side lengths of ΔDEF are longer than those of ΔABC. So, are these triangles similar? If they are, the corresponding sides should be proportional.

Since these triangles are oriented in the same way, you can pair the left, right, and bottom sides: AB¯¯¯¯¯¯¯¯AB¯ and DE¯¯¯¯¯¯¯¯DE¯, BC¯¯¯¯¯¯¯¯BC¯ and EF¯¯¯¯¯¯¯¯EF¯, AC¯¯¯¯¯¯¯¯AC¯ and DF¯¯¯¯¯¯¯¯DF¯. (You might call these the two shortest sides, the two longest sides, and the two leftover sides and arrive at the same ratios). Now we will look at the ratios of their lengths.

AB¯¯¯¯¯¯¯¯DE¯¯¯¯¯¯¯¯=BC¯¯¯¯¯¯¯¯EF¯¯¯¯¯¯¯¯=AC¯¯¯¯¯¯¯¯DF¯¯¯¯¯¯¯¯AB¯DE¯=BC¯EF¯=AC¯DF¯

Substituting the side length values into the proportion, you see that it is true:

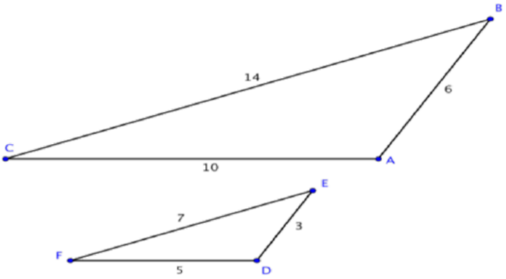
39=412=61839=412=618

If the corresponding sides are proportional, then the triangles are similar. Triangles ABC and DEF are similar, but not congruent.

Let’s use this idea of proportional corresponding sides to determine whether two more triangles are similar.

Example

Determine if the triangles below are similar by seeing if their corresponding sides are proportional.



Solution

First determine the corresponding sides, which are opposite corresponding angles.

CA CA¯ ↔ FD FD¯
AB AB¯ ↔ DE DE¯
BC BC¯ ↔ EF EF¯

Write the corresponding side lengths as ratios.

CA FD =AB DE =B
C EF CA¯FD¯=AB¯DE¯=BC¯EF¯

Substitute the side lengths into the ratios, and determine if the ratios of the corresponding sides are equivalent. They are, so the triangles are similar.

105=63=147

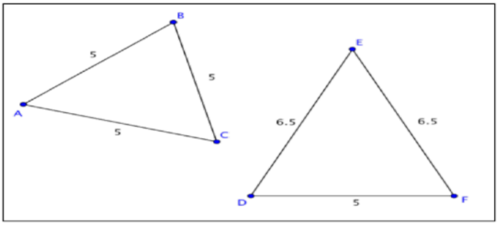
2=2=2

Answer: ΔABC and ΔDEF are similar.

The mathematical symbol ~ means “is similar to”. So, you can write ΔABC is similar to ΔDEF as ΔABC ~ ΔDEF.

Try It Now 2

Determine whether the two triangles are similar, congruent, or neither.

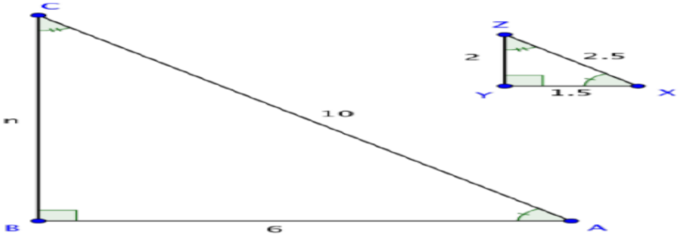


Finding Missing Measurements in Similar Triangles

You can find the missing measurements in a triangle if you know some measurements of a similar triangle. Let’s look at an example.

Example

ΔABC and ΔXYZ are similar triangles. What is the length of side BC?



Solution

In similar triangles, the ratios of corresponding sides are proportional. Set up a proportion of two ratios, one that includes the missing side.

BCYZ=ABXY

Substitute in the known side lengths for the side names in the ratio. Let the unknown side length be n.

$n^2=61.5$

Solve for n using cross multiplication.

$2\cdot 6=1.5\cdot n$

$12=1.5n$

$8=n$

This process is fairly straightforward—but be careful that your ratios represent corresponding sides, recalling that corresponding sides are opposite corresponding angles.

Solving Application Problems Involving Similar Triangles

Applying knowledge of triangles, similarity, and congruence can be very useful for solving problems in real life. Just as you can solve for missing lengths of a triangle drawn on a page, you can use triangles to find unknown distances between locations or objects.

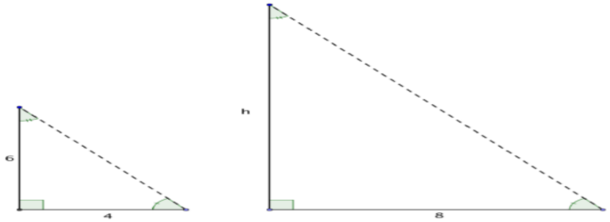
Let’s consider the example of two trees and their shadows. Suppose the sun is shining down on two trees, one that is 6 feet tall and the other whose height is unknown. By measuring the length of each shadow on the ground, you can use triangle similarity to find the unknown height of the second tree.

First, let’s figure out where the triangles are in this situation. The trees themselves create one pair of corresponding sides. The shadows cast on the ground are another pair of corresponding sides. The third side of these imaginary similar triangles runs from the top of each tree to the tip of its shadow on the ground. This is the hypotenuse of the triangle.

If you know that the trees and their shadows form similar triangles, you can set up a proportion to find the height of the tree.

Example

When the sun is at a certain angle in the sky, a 6-foot tree will cast a 4-foot shadow. How tall is a tree that casts an 8-foot shadow?



Solution

The angle measurements are the same, so the triangles are similar triangles. Since they are similar triangles, you can use proportions to find the size of the missing side.

Tree 1Tree 2=Shadow 1Shadow 2

Set up a proportion comparing the heights of the trees and the lengths of their shadows.

Substitute in the known lengths. Call the missing tree height h.

$6h=48$ Solve for h using cross-multiplication.

$6\cdot 8=4h$

$48=4h$

$12=h$

Answer: The tree is 12 feet tall.

Triangles are one of the basic shapes in the real world. Triangles can be classified by the characteristics of their angles and sides, and triangles can be compared based on these characteristics. The sum of the measures of the interior angles of any triangle is 180°. Congruent triangles are triangles of the same size and shape. They have corresponding sides of equal length and corresponding angles of the same measurement. Similar triangles have the same shape, but not necessarily the same size. The lengths of their sides are proportional. Knowledge of triangles can be a helpful in solving real-world problems.

Try It Now Answers

1. Obtuse scalene; this triangle has vertices P, Q, and R, one angle (angle Q) that is between 90° and 180°, and sides of three different lengths.

2. ΔABC and ΔDEF are neither similar nor congruent; the corresponding angle measures are not known to be equal as shown by the absence of congruence marks on the angles. Also, the ratios of the corresponding sides are not equal: $6.55=6.55\neq55$

Pythagorean Theorem

Learning Objectives

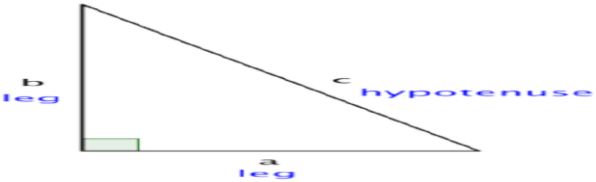
1. Use the Pythagorean Theorem to find the unknown side of a right triangle.
2. Solve application problems involving the Pythagorean Theorem.

A long time ago, a Greek mathematician named **Pythagoras** discovered an interesting property about **right triangles**: the sum of the squares of the lengths of each of the triangle's **legs** is the same as the square of the length of the triangle's **hypotenuse**. This property—which has many applications in science, art, engineering, and architecture—is now called the **Pythagorean Theorem**.

Let's take a look at how this theorem can help you learn more about the construction of triangles. And the best part—you don't even have to speak Greek to apply Pythagoras' discovery.

The Pythagorean Theorem

Pythagoras studied right triangles, and the relationships between the legs and the hypotenuse of a right triangle, before deriving his theory.



The Pythagorean Theorem

If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.

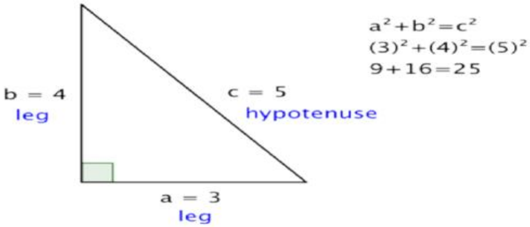
This relationship is represented by the formula: $a^2+b^2=c^2$

In the box above, you may have noticed the word “square,” as well as the small 2s to the top right of the letters in $a^2+b^2=c^2$. To square a number means to multiply it by itself. So, for example, to square the number 5 you multiply 5·5, and to square the number 12, you multiply 12·12. Some common squares are shown in the table below.

Number	**Number Times Itself**	**SQUARE ROOT Y**
1	$1^2=1$	1
2	$2^2=2\cdot 2$	4
3	$3^2=3\cdot 3$	9
4	42=4·4	16
5	$5^2=5\cdot 5$	25
10	$10^2=10\cdot 10$	100

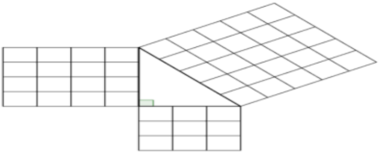
When you see the equation a2+b2=c2a2+b2=c2, you can think of this as “the length of side a times itself, plus the length of side b times itself is the same as the length of side c times itself.”

Let’s try out all of the Pythagorean Theorem with an actual right triangle.



This theorem holds true for this right triangle—the sum of the squares of the lengths of both legs is the same as the square of the length of the hypotenuse. And, in fact, it holds true for all right triangles.

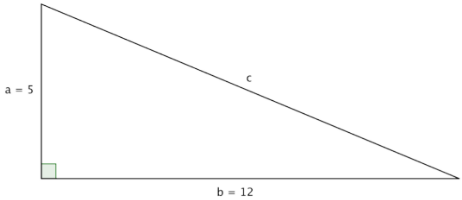
The Pythagorean Theorem can also be represented in terms of area. In any right triangle, the area of the square drawn from the hypotenuse is equal to the sum of the areas of the squares that are drawn from the two legs. You can see this illustrated below in the same 3-4-5 right triangle.



Note that the Pythagorean Theorem only works with right triangles.

Finding the Length of the Hypotenuse

You can use the Pythagorean Theorem to find the length of the hypotenuse of a right triangle if you know the length of the triangle's other two sides, called the legs. Put another way, if you know the lengths of a and b, you can find c.



In the triangle above, you are given measures for legs a and b: 5 and 12, respectively. You can use the Pythagorean Theorem to find a value for the length of c, the hypotenuse.

The Pythagorean Theorem.

$a^2+b^2=c^2$

Substitute known values for a and b.

$(5)^2+(12)^2=c^2$

Evaluate.

$25+144=c^2$

Simplify. To find the value of c, think about a number that, when multiplied by itself, equals 169. Does 10 work? How about 11? 12? 13? (You can use a calculator to multiply if the numbers are unfamiliar.)

$169=c^2$

The square root of 169 is 13.

c=13c=13

Using the formula, you find that the length of c, the hypotenuse, is 13.

In this case, you did not know the value of c—you were given the square of the length of the hypotenuse, and had to figure it out from there. When you are given an equation like 169=c2169=c2 and are asked to find the value of c, this is called finding the **square root** of a number. (Notice you found a number, c, whose square was 169.)

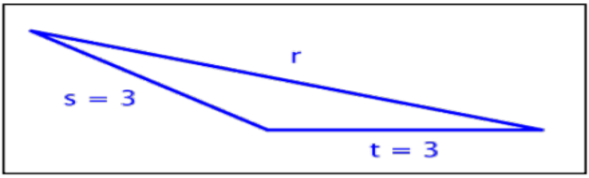
Finding a square root takes some practice, but it also takes knowledge of multiplication, division, and a little bit of trial and error. Look at the table below.

Number x	**Number** y **which, when multiplied by itself, equals number** x	**Square Root** y
1	$1 \cdot 1$	1
4	$2 \cdot 2$	2
9	$3 \cdot 3$	3

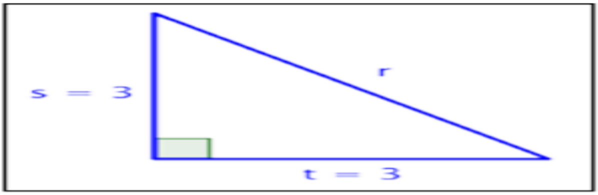
16	4·4	4
25	5·5	5
100	10·10	10

It is a good habit to become familiar with the squares of the numbers from 0–10, as these arise frequently in mathematics. If you can remember those square numbers—or if you can use a calculator to find them—then finding many common square roots will be just a matter of recall.

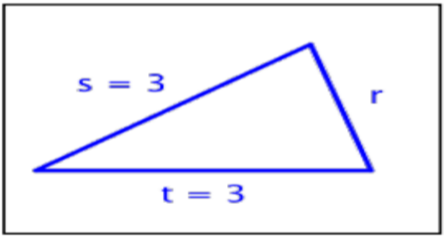
For which of these triangles is $(3)^2+(3)^2=r^2$?

A) 

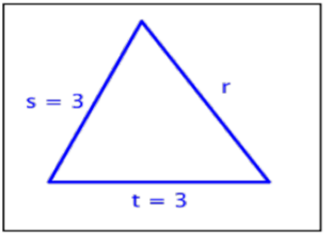
B)



C)



D)

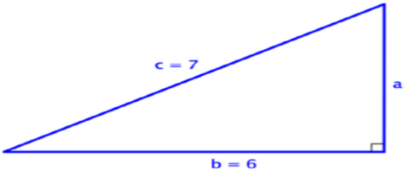


Finding the Length of a Leg

You can use the same formula to find the length of a right triangle's leg if you are given measurements for the lengths of the hypotenuse and the other leg. Consider the example below.

Example

Find the length of side a in the triangle below. Use a calculator to estimate the square root to one decimal place.



Solution

In this right triangle, you are given the measurements for the hypotenuse, c, and one leg, b. The hypotenuse is always opposite the right angle and it is always the longest side of the triangle.

a = ?

b = 6

c = 7

To find the length of leg a, substitute the known values into the Pythagorean Theorem.

$a^2+b^2=c^2$

$a^2+6^2=7^2$

Solve for a^2. Think: what number, when added to 36, gives you 49?

$a^2+36=49$

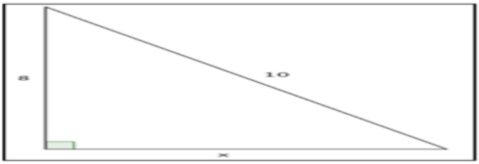
$a^2=13$

Use a calculator to find the square root of 13. The calculator gives an answer of 3.6055…, which you can round to 3.6. (Since you are approximating, you use the symbol ≈.)

$a\approx3.6$

Answer: $a \approx 3.6$

Which of the following correctly uses the Pythagorean Theorem to find the missing side, xx?



A) $8^2+10^2=x^2$

B) $x+8=10$

C) $x^2+8^2=10^2$

D) $x^2+10^2=8^2$

Using the Pythagorean Theorem to Solve Real-world Problems

The Pythagorean Theorem is perhaps one of the most useful formulas you will learn in mathematics because there are so many applications of it in real world settings. Architects and engineers use this formula extensively when building ramps, bridges, and buildings. Look at the following examples.

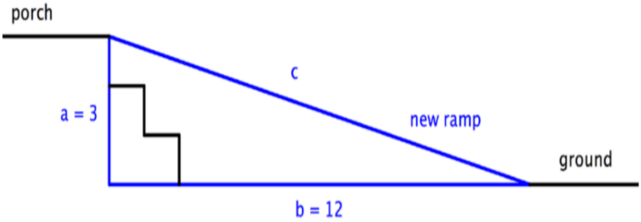
Example

The owners of a house want to convert a stairway leading from the ground to their back porch into a ramp. The porch is 3 feet off the ground, and due to building regulations, the ramp must start 12 feet away from the base of the porch. How long will the ramp be?

Use a calculator to find the square root, and round the answer to the nearest tenth.

Solution

To solve a problem like this one, it often makes sense to draw a simple diagram showing where the legs and hypotenuse of the triangle lie.



Identify the legs and the hypotenuse of the triangle. You know that the triangle is a right triangle since the ground and the raised portion of the porch are perpendicular—this means you can use the Pythagorean Theorem to solve this problem. Identify a, b, and c.

$a = 3$

$b = 12$

$c = ?$

Use the Pythagorean Theorem to find the length of c.

$a^2+b^2=c^2$

$3^2+12^2=c^2$

$9+144=c^2$

$153=c^2$

Use a calculator to find c.

$12.4=c^2$

The square root of 153 is 12.369…, so you can round that to 12.4.

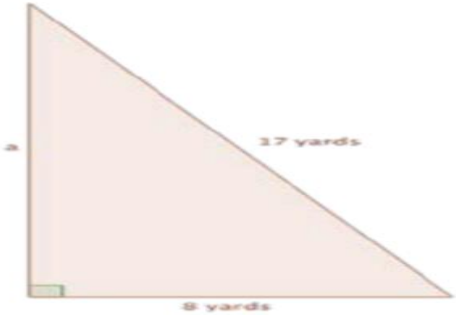
Answer: The ramp will be 12.4 feet long.

Example

A sailboat has a large sail in the shape of a right triangle. The longest edge of the sail measures 17 yards, and the bottom edge of the sail is 8 yards. How tall is the sail?

Solution

Draw an image to help you visualize the problem. In a right triangle, the hypotenuse will always be the longest side, so here it must be 17 yards. The problem also tells you that the bottom edge of the triangle is 8 yards.



Setup the Pythagorean Theorem.

$a^2+b^2=c^2$

$a^2+8^2=17^2$

$a^2+64=289$

$a^2=225$

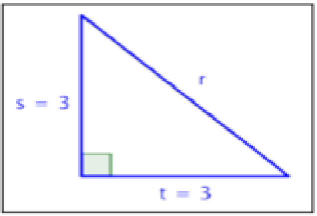
$15\cdot15=225$, so

$a=15$

Answer: The height of the sail is 15 yards.

The Pythagorean Theorem states that in any right triangle, the sum of the squares of the lengths of the triangle’s legs is the same as the square of the length of the triangle’s hypotenuse. This theorem is represented by the formula $a^2+b^2=c^2$. Put simply, if you know the lengths of two sides of a right triangle, you can apply the Pythagorean Theorem to find the length of the third side. Remember, this theorem only works for right triangles.

1. B) ; this is a right triangle; when you sum the



squares of the lengths of the sides, you get the square of the length of the hypotenuse.

2. C) $x^2+8^2=102$; in this triangle, the hypotenuse has length 10, and the legs have length 8 and x. Substituting into the Pythagorean Theorem you have: $x^2+8^2=10^2$; this equation is the same as $x^2+64=100$, or $x2=36$. What number, times itself, equals 36? That would make $x=6$.

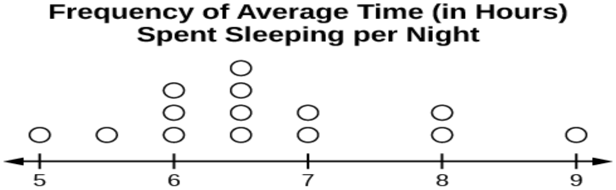
§13. Statistical regularities, signs and their classification. Forms of presentation of statistical data: table, column and pie charts (telegraph, bargraph, pictogram), time-varying linear graph, etc. Concepts of average result (median), most common result (mode), largest difference, mean statistical result

Collaborative exercise

In your classroom, try this exercise. Have class members write down the average time (in hours, to the nearest half-hour) they sleep per night. Your instructor will record the data. Then create a simple graph (called a **dot plot**) of the data. A dot plot consists of a number line and dots (or points) positioned above the number line. For example, consider the following data:

5; 5.5; 6; 6; 6; 6.5; 6.5; 6.5; 6.5; 7; 7; 8; 8; 9

The dot plot for this data would be as follows:



Figure

Does your dot plot look the same as or different from the example? Why? If you did the same example in an English class with the same number of students, do you think the results would be the same? Why or why not?

Where do your data appear to cluster? How might you interpret the clustering?

The questions above ask you to analyze and interpret your data. With this example, you have begun your study of statistics. In this course, you will learn how to organize and summarize data. Organizing and summarizing data is called **descriptive statistics**. Two ways to summarize data are by graphing and by using numbers (for example, finding an average). After you have studied probability and probability distributions, you will use formal methods for drawing conclusions from "good" data. The formal methods are called **inferential statistics**. Statistical inference uses probability to determine how confident we can be that our conclusions are correct.

Effective interpretation of data (inference) is based on good procedures for producing data and thoughtful examination of the data. You will encounter what will seem to be too many mathematical formulas for interpreting data. The goal of statistics is not to perform numerous calculations using the formulas, but to gain an understanding of your data. The calculations can be done using a calculator or a computer. The understanding must come from you. If you can thoroughly grasp the basics of statistics, you can be more confident in the decisions you make in life.

Probability

Probability is a mathematical tool used to study randomness. It deals with the chance (the likelihood) of an event occurring. For example, if you toss a fair coin four times, the outcomes may not be two heads and two tails. However, if you toss the same coin 4,000 times, the outcomes will be close to half heads and half tails. The expected theoretical probability of heads in any one toss is 1212 or 0.5. Even though the outcomes of a few repetitions are uncertain, there is a regular pattern of outcomes when there are many repetitions. After reading about the English statistician Karl Pearson who tossed a coin 24,000 times with a result of 12,012 heads, one of the authors tossed a coin 2,000 times.

The results were 996 heads. The fraction 99620009962000 is equal to 0.498 which is very close to 0.5, the expected probability. 12 or 0.5. Even though the outcomes of a few repetitions are uncertain, there is a regular pattern of outcomes when there are many repetitions. After reading about the English statistician Karl Pearson who tossed a coin 24,000 times with a result of 12,012 heads, one of the authors tossed a coin 2,000 times. The results were 996 heads. The fraction 99620009962000 is equal to 0.498 which is very close to 0.5, the expected probability.

The theory of probability began with the study of games of chance such as poker. Predictions take the form of probabilities. To predict the likelihood of an earthquake, of rain, or whether you will get an A in this course, we use probabilities. Doctors use probability to determine the chance of a vaccination causing the disease the vaccination is supposed to prevent. A stockbroker uses probability to determine the rate of return on a client's investments. You might use probability to decide to buy a lottery ticket or not. In your study of statistics, you will use the power of mathematics through probability calculations to analyze and interpret your data.

Key Terms

In statistics, we generally want to study a **population**. You can think of a population as a collection of persons, things, or objects under study. To study the population, we select a **sample**. The idea of **sampling** is to select a portion (or subset) of the larger population and study that portion (the sample) to gain information about the population. Data are the result of sampling from a population.

Because it takes a lot of time and money to examine an entire population, sampling is a very practical technique. If you wished to compute the overall grade point average at your school, it would make sense to select a sample of students who attend the school. The data collected from the sample would be the students' grade point averages. In presidential elections, opinion poll samples of 1,000–2,000 people are taken. The opinion poll is supposed to represent the views of the people in the entire country. Manufacturers of canned carbonated drinks take samples to determine if a 16 ounce can contains 16 ounces of carbonated drink.

From the sample data, we can calculate a statistic. A **statistic** is a number that represents a property of the sample. For example, if we consider one math class to be a sample of the population of all math classes, then the average number of points earned by students in that one math class at the end of the term is an example of a statistic. The statistic is an estimate of a population parameter. A **parameter** is a number that is a property of the population. Since we considered all math classes to be the population, then the average number of points earned per student over all the math classes is an example of a parameter.

One of the main concerns in the field of statistics is how accurately a statistic estimates a parameter. The accuracy really depends on how well the sample represents the population. The sample must contain the characteristics of the population in order to be a **representative sample**. We are interested in both the sample statistic and the population parameter in inferential statistics. In a later chapter, we will use the sample statistic to test the validity of the established population parameter.

A **variable**, notated by capital letters such as XX and YY, is a characteristic of interest for each person or thing in a population. Variables may be **numerical** or **categorical**. **Numerical variables** take on values with equal units such as weight in pounds and time in hours. **Categorical variables** place the person or thing into a category. If we let XX equal the number of points earned by one math student at the end of a term, then XX is a numerical variable. If we let YY be a person's party affiliation, then some examples of YY include Republican, Democrat, and Independent. YY is a categorical variable. We could do some math with values of XX (calculate the average number of points earned, for example), but it makes no sense to do math with values of YY (calculating an average party affiliation makes no sense).

Data are the actual values of the variable. They may be numbers or they may be words. **Datum** is a single value. Two words that come up often in statistics are **mean** and **proportion**. If you were to take three exams in your math classes and obtain scores of 86, 75, and 92, you would calculate your mean score by adding the three exam scores and dividing by three (your mean score would be 84.3 to one decimal place). If, in your math class, there are 40 students and 22 are men and 18 are women, then the proportion of men students is 22402240 and the proportion of women students is 18401840. Mean and proportion are discussed in more detail in later chapters.

The words "**mean**" and "**average**" are often used interchangeably. The substitution of one word for the other is common practice. The technical term is "arithmetic mean," and "average" is technically a center location. However, in practice among non-statisticians, "average" is commonly accepted for "arithmetic mean."

Example

Determine what the key terms refer to in the following study. We want to know the average (mean) amount of money first year college students spend at ABC College on school supplies that do not include books. We randomly survey 100 first year students at the college. Three of those students spent $150, $200, and $225, respectively.

Answer

- The **population** is all first year students attending ABC College this term.
- The **sample** could be all students enrolled in one section of a beginning statistics course at ABC College (although this sample may not represent the entire population).
- The **parameter** is the average (mean) amount of money spent (excluding books) by first year college students at ABC College this term.
- The **statistic** is the average (mean) amount of money spent (excluding books) by first year college students in the sample.
- The **variable** could be the amount of money spent (excluding books) by one first year student. Let XX = the amount of money spent (excluding books) by one first year student attending ABC College.
- The **data** are the dollar amounts spent by the first year students. Examples of the data are $150, $200, and $225.

Exercise

Determine what the key terms refer to in the following study. We want to know the average (mean) amount of money spent on school uniforms each year by families with children at Knoll Academy. We randomly survey 100 families with children in the school. Three of the families spent $65, $75, and $95, respectively.

Answer

- The **population** is all families with children attending Knoll Academy.

- The **sample** is a random selection of 100 families with children attending Knoll Academy.
- The **parameter** is the average (mean) amount of money spent on school uniforms by families with children at Knoll Academy.
- The **statistic** is the average (mean) amount of money spent on school uniforms by families in the sample.
- The **variable** is the amount of money spent by one family. Let XX = the amount of money spent on school uniforms by one family with children attending Knoll Academy.
- The **data** are the dollar amounts spent by the families. Examples of the data are \$65, \$75, and \$95.

Example

Determine what the key terms refer to in the following study. A study was conducted at a local college to analyze the average cumulative GPA's of students who graduated last year. Fill in the letter of the phrase that best describes each of the items below.

1._____ Population
2._____ Statistic
3._____ Parameter
4._____ Sample
5._____ Variable
6._____ Data

a. all students who attended the college last year

b. the cumulative GPA of one student who graduated from the college last year

c. 3.65, 2.80, 1.50, 3.90

d. a group of students who graduated from the college last year, randomly selected

e. the average cumulative GPA of students who graduated from the college last year

f. all students who graduated from the college last year

g. the average cumulative GPA of students in the study who graduated from the college last year

Answer

1. f; 2. g; 3. e; 4. d; 5. b; 6. c

Example

Determine what the key terms refer to in the following study. As part of a study designed to test the safety of automobiles, the National Transportation Safety Board collected and reviewed data about the effects of an automobile crash on test dummies. Here is the criterion they used:

Speed at which Cars Crashed	Location of “drive” (i.e. dummies)
35 miles/hour	Front Seat

Cars with dummies in the front seats were crashed into a wall at a speed of 35 miles per hour. We want to know the proportion of dummies in the driver’s seat that would have had head injuries, if they had been actual drivers. We start with a simple random sample of 75 cars.

Answer

- The **population** is all cars containing dummies in the front seat.
- The **sample** is the 75 cars, selected by a simple random sample.

- The **parameter** is the proportion of driver dummies (if they had been real people) who would have suffered head injuries in the population.
- The **statistic** is proportion of driver dummies (if they had been real people) who would have suffered head injuries in the sample.
- The **variable** XX = the number of driver dummies (if they had been real people) who would have suffered head injuries.
- The **data** are either: yes, had head injury, or no, did not.

Example

Determine what the key terms refer to in the following study. An insurance company would like to determine the proportion of all medical doctors who have been involved in one or more malpractice lawsuits. The company selects 500 doctors at random from a professional directory and determines the number in the sample who have been involved in a malpractice lawsuit.

Answer

- The **population** is all medical doctors listed in the professional directory.
- The **parameter** is the proportion of medical doctors who have been involved in one or more malpractice suits in the population.
- The **sample** is the 500 doctors selected at random from the professional directory.
- The **statistic** is the proportion of medical doctors who have been involved in one or more malpractice suits in the sample.
- The **variable** XX = the number of medical doctors who have been involved in one or more malpractice suits.

- The **data** are either: yes, was involved in one or more malpractice lawsuits, or no, was not.

COLLABORATIVE EXERCISE

Do the following exercise collaboratively with up to four people per group. Find a population, a sample, the parameter, the statistic, a variable, and data for the following study: You want to determine the average (mean) number of glasses of milk college students drink per day. Suppose yesterday, in your English class, you asked five students how many glasses of milk they drank the day before. The answers were 1, 0, 1, 3, and 4 glasses of milk.

References

1. The Data and Story Library, http://lib.stat.cmu.edu/DASL/Stories...stDummies.html (accessed May 1, 2013).

Practice

Use the following information to answer the next five exercises. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average (mean) length of time in months patients live once they start the treatment. Two researchers each follow a different set of 40 patients with AIDS from the start of treatment until their deaths. The following data (in months) are collected.

Researcher A:

3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34

Researcher B:

3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29

Determine what the key terms refer to in the example for Researcher A.

Exercise

population

Answer

AIDS patients.

Exercise

sample

Exercise

parameter

Answer

The average length of time (in months) AIDS patients live after treatment.

Exercise

statistic

Exercise

variable

Answer

$X=X=$ the length of time (in months) AIDS patients live after treatment

Concepts of average result (median), most common result (mode), largest difference, mean statistical result

The terms mean, median, mode, and range describe properties of statistical distributions. In statistics, a distribution is the set of all possible values for terms that represent defined events.

The value of a term, when expressed as a variable, is called a random variable.

There are two major types of statistical distributions. The first type contains discrete random variables. This means that every term has a precise, isolated numerical value. The second major type of distribution contains a continuous random variable. A continuous random variable is a random variable where the data can take infinitely many values. When a term can acquire any value within an unbroken interval or span, it is called a probability density function.

IT professionals need to understand the definition of mean, median, mode and range to plan capacity and balance load, manage systems, perform maintenance and troubleshoot issues. Furthermore, understanding of statistical terms is important in the growing field of data science.

How are mean, median, mode and range used in the data center?

Understanding the definition of mean, median, mode and range is important for IT professionals in data center management. Many relevant tasks require the administrator to calculate mean, median, mode or range, or often some combination, to show a statistically significant quantity, trend or deviation from the norm. Finding the mean, median, mode and range is only the start. The administrator then needs to apply this information to investigate root causes of a problem, accurately forecast future needs or set acceptable working parameters for IT systems.

When working with a large data set, it can be useful to represent the entire data set with a single value that describes the "middle" or "average" value of the entire set. In statistics, that single value is called the central tendency and mean, median and mode are all ways to describe it. To find the mean, add up the values in the data set and then divide by the number of values that you added. To find the median, list the values of the data set in numerical order and identify which value appears in the middle of the list. To find the mode, identify which value in the data set occurs most often. Range, which is the difference between the largest and smallest value in the data set, describes how well the central tendency represents the data. If the range is large, the central tendency is not as representative of the data as it would be if the range was small.

Mean

The most common expression for the mean of a statistical distribution with a discrete random variable is the mathematical average of all the terms. To calculate it, add up the values of all the terms and then divide by the number of terms. The mean of a statistical distribution with a continuous random variable, also called the expected value, is obtained by integrating the product of the variable with its probability as defined by the distribution. The expected value is denoted by the lowercase Greek letter mu (μ).

Median

The median of a distribution with a discrete random variable depends on whether the number of terms in the distribution is even or odd. If the number of terms is odd, then the median is the value of the term in the middle. This is the value such that the number of terms having values greater than or equal to it is the same as the number of terms having values less than or equal to it. If the number of terms is even, then the median is the average of the two terms in the middle, such that the number of terms having values greater than or equal to it is the same as the number of terms having values less than or equal to it.

The median of a distribution with a continuous random variable is the value m such that the probability is at least 1/2 (50%) that a randomly chosen point on the function will be less than or equal to m, and the probability is at least 1/2 that a randomly chosen point on the function will be greater than or equal to m.

Mode

The mode of a distribution with a discrete random variable is the value of the term that occurs the most often. It is not uncommon for a distribution with a discrete random variable to have more than one mode, especially if there are not many terms. This happens when two or more terms occur with equal frequency, and more often than any of the others.

A distribution with two modes is called bimodal. A distribution with three modes is called trimodal. The mode of a distribution with a continuous random variable is the maximum value of the function. As with discrete distributions, there may be more than one mode.

Range

The range of a distribution with a discrete random variable is the difference between the maximum value and the minimum value. For a distribution with a continuous random variable, the range is the difference between the two extreme points on the distribution curve, where the value of the function falls to zero. For any value outside the range of a distribution, the value of the function is equal to 0.

Using mean to determine power usage

To calculate mean, add together all of the numbers in a set and then divide the sum by the total count of numbers. For example, in a data center rack, five servers consume 100 watts, 98 watts, 105 watts, 90 watts and 102 watts of power, respectively. The mean power use of that rack is calculated as (100 + 98 + 105 + 90 + 102 W)/5 servers = a calculated mean of 99 W per server. Intelligent power distribution units report the mean power utilization of the rack to systems management software.

Using median to plan capacity

In the data center, means and medians are often tracked over time to spot trends, which inform capacity planning or power cost predictions.The statistical median is the middle number in a sequence of numbers. To find the median, organize each number in order by size; the number in the middle is the median. For the five servers in the rack, arrange the power consumption figures from lowest to highest: 90 W, 98 W, 100 W, 102 W and 105 W. The median power consumption of the rack is 100 W. If there is an even set of numbers, average the two middle numbers. For example, if the rack had a sixth server that used 110 W, the new number set would be 90 W, 98 W, 100 W, 102 W, 105 W and 110 W. Find the median by averaging the two middle numbers: (100 + 102)/2 = 101 W.

Using mode to identify a base line

The mode is the number that occurs most often within a set of numbers. For the server power consumption examples above, there is no mode because each element is different. But suppose the administrator measured the power consumption of an entire network operations center (NOC) and the set of numbers is 90 W, 104 W, 98 W, 98 W, 105 W, 92 W, 102 W, 100 W, 110 W, 98 W, 210 W and 115 W. The mode is 98 W since that power consumption measurement occurs most often amongst the 12 servers. Mode helps identify the most common or frequent occurrence of a characteristic. It is possible to have two modes (bimodal), three modes (trimodal) or more modes within larger sets of numbers.

Using range to identify outliers

The range is the difference between the highest and lowest values within a set of numbers. To calculate range, subtract the smallest number from the largest number in the set. If a six-server rack includes 90 W, 98 W, 100 W, 102 W, 105 W and 110 W, the power consumption range is 110 W - 90 W = 20 W.

Range shows how much the numbers in a set vary. Many IT systems operate within an acceptable range; a value in excess of that range might trigger a warning or alarm to IT staff. To find the variance in a data set, subtract each number from the mean, and then square the result. Find the average of these squared differences, and that is the variance in the group. In our original group of five servers, the mean was 99. The 100 W-server varies from the mean by 1 W, the 105 W-server by 6 W, and so on. The squares of each difference equal 1, 1, 36, 81 and 9. So to calculate the variance, add 1 + 1 + 36 + 81 + 9 and divide by 5. The variance is 25.6. Standard deviation denotes how far apart all the numbers are in a set. The standard deviation is calculated by finding the square root of the variance. In this example, the standard deviation is 5.1.

Interquartile range, the middle fifty or midspread of a set of numbers, removes the outliers -- highest and lowest numbers in a set. If there is a large set of numbers, divide them evenly into lower and higher numbers. Then find the median of each of these groups. Find the interquartile range by subtracting the lower median from the higher median. If a rack of six servers' power wattage is arranged from lowest to highest: 90, 98, 100, 102, 105, 110, divide this set into low numbers (90, 98, 100) and high numbers (102, 105, 110). Find the median for each: 98 and 105. Subtract the lower median from the higher median: 105 watts - 98 W = 7 W, which is the interquartile range of these servers.

§14. The concept of probability. Experiment, classification of events: "Probable", "possible" and "impossible" events. The classic definition of probability

The **classical definition** or **interpretation of probability** is identified with the works of Jacob Bernoulli and Pierre-Simon Laplace. As stated in Laplace's Théorie analytique des probabilités,

The probability of an event is the ratio of the number of cases favorable to it, to the number of all cases possible when nothing leads us to expect that any one of these cases should occur more than any other, which renders them, for us, equally possible.

This definition is essentially a consequence of the principle of indifference. If elementary events are assigned equal probabilities, then the probability of a disjunction of elementary events is just the number of events in the disjunction divided by the total number of elementary events.

The classical definition of probability was called into question by several writers of the nineteenth century, including John Venn and George Boole. The frequentist definition of probability became widely accepted as a result of their criticism, and especially through the works of R.A. Fisher. The classical definition enjoyed a revival of sorts due to the general interest in Bayesian probability, because Bayesian methods require a prior probability distribution and the principle of indifference offers one source of such a distribution. Classical probability can offer prior probabilities that reflect ignorance which often seems appropriate before an experiment is conducted.

Investment decisions are made in a risky environment. The tools that allow us to make decisions with consistency and logic in this setting are based on probability concepts. This reading presents the essential probability tools needed to frame and address many real-world problems involving risk. These tools apply to a variety of issues, such as predicting investment manager performance, forecasting financial variables, and pricing bonds so that they fairly compensate bondholders for default risk. Our focus is practical. We explore the concepts that are most important to investment research and practice. Among these are independence, as it relates to the predictability of returns and financial variables; expectation, as analysts continually look to the future in their analyses and decisions; and variability, variance or dispersion around expectation, as a risk concept important in investments. The reader will acquire specific skills and competencies in using these probability concepts to understand risks and returns on investments.

Learning Outcomes

The member should be able to:

- define a random variable, an outcome, and an event;
- identify the two defining properties of probability, including mutually exclusive and exhaustive events, and compare and contrast empirical, subjective, and a priori probabilities;
- describe the probability of an event in terms of odds for and against the event;
- calculate and interpret conditional probabilities;
- demonstrate the application of the multiplication and addition rules for probability;
- compare and contrast dependent and independent events;

- calculate and interpret an unconditional probability using the total probability rule;
- calculate and interpret the expected value, variance, and standard deviation of random variables;
- explain the use of conditional expectation in investment applications;
- interpret a probability tree and demonstrate its application to investment problems;
- calculate and interpret the expected value, variance, standard deviation, covariances, and correlations of portfolio returns;
- calculate and interpret the covariances of portfolio returns using the joint probability function;
- calculate and interpret an updated probability using Bayes' formula;
- identify the most appropriate method to solve a particular counting problem and analyze counting problems using factorial, combination, and permutation concepts.

Summary

In this reading, we have discussed the essential concepts and tools of probability. We have applied probability, expected value, and variance to a range of investment problems.

- A random variable is a quantity whose outcome is uncertain.
- Probability is a number between 0 and 1 that describes the chance that a stated event will occur.
- An event is a specified set of outcomes of a random variable.
- Mutually exclusive events can occur only one at a time. Exhaustive events cover or contain all possible outcomes.

- The two defining properties of a probability are, first, that $0 \leq P(E) \leq 1$ (where P(E) denotes the probability of an event E), and second, that the sum of the probabilities of any set of mutually exclusive and exhaustive events equals 1.
- A probability estimated from data as a relative frequency of occurrence is an empirical probability. A probability drawing on personal or subjective judgment is a subjective probability. A probability obtained based on logical analysis is an a priori probability.
- A probability of an event E, P(E), can be stated as odds for $E = P(E)/[1 - P(E)]$ or odds against $E = [1 - P(E)]/P(E)$.
- Probabilities that are inconsistent create profit opportunities, according to the Dutch Book Theorem.
- A probability of an event not conditioned on another event is an unconditional probability. The unconditional probability of an event A is denoted P(A). Unconditional probabilities are also called marginal probabilities.
- A probability of an event given (conditioned on) another event is a conditional probability. The probability of an event A given an event B is denoted $P(A \mid B)$, and $P(A \mid B) = P(AB)/P(B)$, $P(B) \neq 0$.
- The probability of both A and B occurring is the joint probability of A and B, denoted P(AB).
- The multiplication rule for probabilities is $P(AB) = P(A \mid B)P(B)$.
- The probability that A or B occurs, or both occur, is denoted by P(A or B).
- The addition rule for probabilities is $P(A \text{ or } B) = P(A) + P(B) - P(AB)$.

- When events are independent, the occurrence of one event does not affect the probability of occurrence of the other event. Otherwise, the events are dependent.
- The multiplication rule for independent events states that if A and B are independent events, $P(AB) = P(A)P(B)$. The rule generalizes in similar fashion to more than two events.
- According to the total probability rule, if $S_1, S_2, \ldots, S_n$ are mutually exclusive and exhaustive scenarios or events, then $P(A) = P(A \mid S_1)P(S_1) + P(A \mid S_2)P(S_2) + \ldots + P(A \mid S_n)P(S_n)$.
- The expected value of a random variable is a probability-weighted average of the possible outcomes of the random variable. For a random variable X, the expected value of X is denoted E(X).
- The total probability rule for expected value states that $E(X) = E(X \mid S_1)P(S_1) + E(X \mid S_2)P(S_2) + \ldots + E(X \mid S_n)P(S_n)$, where $S_1, S_2, \ldots, S_n$ are mutually exclusive and exhaustive scenarios or events.
- The variance of a random variable is the expected value (the probability-weighted average) of squared deviations from the random variable's expected value E(X): $\sigma^2(X) = E\{[X - E(X)]^2\}$, where $\sigma^2(X)$ stands for the variance of X.
- Variance is a measure of dispersion about the mean. Increasing variance indicates increasing dispersion. Variance is measured in squared units of the original variable.
- Standard deviation is the positive square root of variance. Standard deviation measures dispersion (as does variance), but it is measured in the same units as the variable.
- Covariance is a measure of the co-movement between random variables.

- The covariance between two random variables R_i and R_j in a forward-looking sense is the expected value of the cross-product of the deviations of the two random variables from their respective means: $\mathrm{Cov}(R_i, R_j) = E\{[R_i - E(R_i)][R_j - E(R_j)]\}$. The covariance of a random variable with itself is its own variance.
- The historical or sample covariance between two random variables R_i and R_j based on a sample of past data of size n is the average value of the product of the deviations of observations on two random variables from their sample means:

Cov(Ri,Rj)=n∑i=1(Ri,t−‾‾Ri)(Rj,t−‾‾Rj)/(n−1)Cov(Ri,Rj) =∑i=1n(Ri,t−R¯i)(Rj,t−R¯j)/(n−1)

- Correlation is a number between −1 and +1 that measures the co-movement (linear association) between two random variables: $\rho(R_i, R_j) = \mathrm{Cov}(R_i, R_j)/[\sigma(R_i)\,\sigma(R_j)]$.
- If two variables have a very strong linear relation, then the absolute value of their correlation will be close to 1. If two variables have a weak linear relation, then the absolute value of their correlation will be close to 0.
- If the correlation coefficient is positive, the two variables are directly related; if the correlation coefficient is negative, the two variables are inversely related.
- To calculate the variance of return on a portfolio of n assets, the inputs needed are the n expected returns on the individual assets, n variances of return on the individual assets, and n(n − 1)/2 distinct covariances.
- Portfolio variance of return is σ2(Rp)=n∑i=1n∑j=1wiwjCov(Ri,Rj)σ2(Rp)=∑i=1n∑j=1n wiwjCov(Ri,Rj).

- The calculation of covariance in a forward-looking sense requires the specification of a joint probability function, which gives the probability of joint occurrences of values of the two random variables.
- When two random variables are independent, the joint probability function is the product of the individual probability functions of the random variables.
- Bayes' formula is a method for updating probabilities based on new information.
- Bayes' formula is expressed as follows: Updated probability of event given the new information = [(Probability of the new information given event)/(Unconditional probability of the new information)] × Prior probability of event.
- The multiplication rule of counting says, for example, that if the first step in a process can be done in 10 ways, the second step, given the first, can be done in 5 ways, and the third step, given the first two, can be done in 7 ways, then the steps can be carried out in (10)(5)(7) = 350 ways.
- The number of ways to assign every member of a group of size n to n slots is $n! = n(n-1)(n-2)(n-3)\ldots 1$. (By convention, $0! = 1$.)
- The number of ways that n objects can be labeled with k different labels, with n_1 of the first type, n_2 of the second type, and so on, with $n_1 + n_2 + \ldots + n_k = n$, is given by $n!/(n_1!n_2! \ldots n_k!)$. This expression is the multinomial formula.
- A special case of the multinomial formula is the combination formula. The number of ways to choose r objects

from a total of n objects, when the order in which the r objects are listed does not matter, is

nCr=(nr)=n!(n−r)!r!nCr=(nr)=n!(n−r)!r!

- The number of ways to choose r objects from a total of n objects, when the order in which the r objects are listed does matter, is

nPr=n!(n−r)!nPr=n!(n−r)!

This expression is the permutation formula.

What's the probability of an impossible event?

Solution:

Randomness can be measured mathematically by using probability. It is defined as the ratio of total number of favourable outcomes to the total number of possible outcomes. Probability of any sure event is always equal to 1 as the total number of favourable outcomes is equal to the total number of possible outcomes. The probability of an impossible event is 0 because it cannot occur in any situation.

In general, probability ranges from 0 to 1.

0 implies that the event is an impossible situation and 1 represents an event to occur for sure.

Example: Getting a number 7 while rolling a die is an impossible event with 0 probability. (Since, a die has numbers ranging from 1 to 6)

Thus, the probability of an impossible event is 0.

Probability

Probability defines the likelihood of occurrence of an event. There are many real-life situations in which we may have to predict the outcome of an event. We may be sure or not sure of the results of an event. In such cases, we say that there is a probability of this event to occur or not occur. Probability generally has great applications in games, in business to make probability-based predictions, and also probability has extensive applications in this new area of artificial intelligence.

The probability of an event can be calculated by probability formula by simply dividing the favorable number of outcomes by the total number of possible outcomes. The value of the probability of an event to happen can lie between 0 and 1 because the favorable number of outcomes can never cross the total number of outcomes. Also, the favorable number of outcomes cannot be negative. Let us discuss the basics of probability in detail in the following sections.

What is Probability?

Probability can be defined as the ratio of the number of favorable outcomes to the total number of outcomes of an event. For an experiment having 'n' number of outcomes, the number of favorable outcomes can be denoted by x. The formula to calculate the probability of an event is as follows.

Probability(Event) = Favorable Outcomes/Total Outcomes = x/n

Let us check a simple application of probability to understand it better. Suppose we have to predict about the happening of rain or not. The answer to this question is either "Yes" or "No". There is a likelihood to rain or not rain. Here we can apply probability. Probability is used to predict the outcomes for the tossing of coins, rolling of dice, or drawing a card from a pack of playing cards.

The probability is classified into theoretical probability and experimental probability.

Terminology of Probability Theory

The following terms in probability help in a better understanding of the concepts of probability.

Experiment: A trial or an operation conducted to produce an outcome is called an experiment.

Sample Space: All the possible outcomes of an experiment together constitute a sample space. For example, the sample space of tossing a coin is head and tail.

Favorable Outcome: An event that has produced the desired result or expected event is called a favorable outcome. For example, when we roll two dice, the possible/favorable outcomes of getting the sum of numbers on the two dice as 4 are (1,3), (2,2), and (3,1).

Trial: A trial denotes doing a random experiment.

Random Experiment: An experiment that has a well-defined set of outcomes is called a random experiment. For example, when we toss a coin, we know that we would get ahead or tail, but we are not sure which one will appear.

Event: The total number of outcomes of a random experiment is called an event.

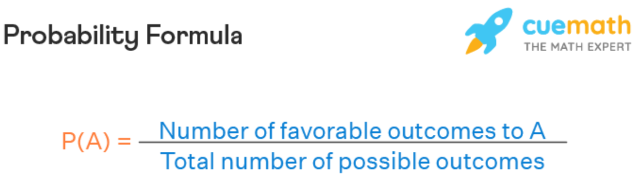
Equally Likely Events: Events that have the same chances or probability of occurring are called equally likely events. The outcome of one event is independent of the other. For example, when we toss a coin, there are equal chances of getting a head or a tail.

Exhaustive Events: When the set of all outcomes of an experiment is equal to the sample space, we call it an exhaustive event.

Mutually Exclusive Events: Events that cannot happen simultaneously are called mutually exclusive events. For example, the climate can be either hot or cold. We cannot experience the same weather simultaneously.

Probability Formula

The probability formula defines the likelihood of the happening of an event. It is the ratio of favorable outcomes to the total favorable outcomes. The probability formula can be expressed as,



Probability Formula

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$$P(A) = \frac{\text{Number of favorable outcomes to A}}{\text{Total number of possible outcomes}}$$

where,

- P(B) is the probability of an event 'B'.
- n(B) is the number of favorable outcomes of an event 'B'.
- n(S) is the total number of events occurring in a sample space.

Different Probability Formulas

Probability formula with addition rule: Whenever an event is the union of two other events, say A and B, then $P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Probability formula with the complementary rule: Whenever an event is the complement of another event, specifically, if A is an event, then $P(\text{not } A) = 1 - P(A)$ or $P(A') = 1 - P(A)$.
$P(A) + P(A') = 1$.

Probability formula with the conditional rule: When event A is already known to have occurred and the probability of event B is desired, then P(B, given A) = P(A and B), P(A, given B). It can be vice versa in the case of event B. $P(B|A) = P(A \cap B)/P(A)$

Probability formula with multiplication rule: Whenever an event is the intersection of two other events, that is, events A and B need to occur simultaneously. Then

P(A and B) = P(A)·P(B). P(A∩B) = P(A)·P(B|A)

Example 1: Find the probability of getting a number less than 5 when a dice is rolled by using the probability formula.

Solution

To find:

Probability of getting a number less than 5

Given: Sample space = {1,2,3,4,5,6}

Getting a number less than 5 = {1,2,3,4}

Therefore, n(S) = 6

n(A) = 4

Using Probability Formula,

P(A) = (n(A))/(n(s))

p(A) = 4/6

m = 2/3

Answer: The probability of getting a number less than 5 is 2/3.

Example 2: What is the probability of getting a sum of 9 when two dice are thrown?

Solution:

There is a total of 36 possibilities when we throw two dice.

To get the desired outcome i.e., 9, we can have the following favorable outcomes.

(4,5),(5,4),(6,3)(3,6). There are 4 favorable outcomes.

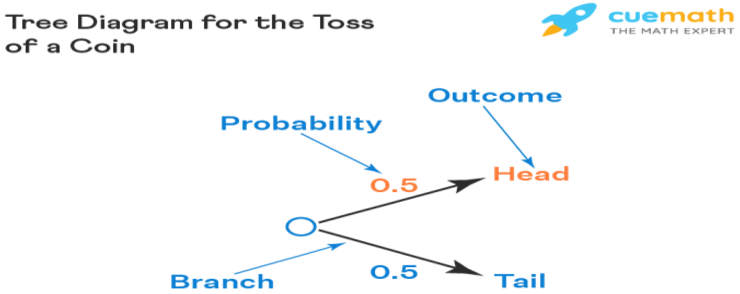
Probability of an event P(E) = (Number of favorable outcomes) ÷ (Total outcomes in a sample space)

Probability of getting number 9 = 4 ÷ 36 = 1/9

Answer: Therefore the probability of getting a sum of 9 is 1/9.

Probability Tree Diagram

A tree diagram in probability is a visual representation that helps in finding the possible outcomes or the probability of any event occurring or not occurring. The tree diagram for the toss of a coin given below helps in understanding the possible outcomes when a coin is tossed and thus in finding the probability of getting a head or tail when a coin is tossed.



Types of Probability

There can be different perspectives or types of probabilities based on the nature of the outcome or the approach followed while finding the probability of an event happening. The four types of probabilities are,

- Classical Probability
- Empirical Probability
- Subjective Probability
- Axiomatic Probability

Classical Probability

Classical probability, often referred to as the "priori" or "theoretical probability", states that in an experiment where there are B equally likely outcomes, and event X has exactly A of these outcomes, then the probability of X is A/B, or P(X) = A/B. For example, when a fair die is rolled, there are six possible outcomes that are equally likely. That means, there is a 1/6 probability of rolling each **number on the die.**

Empirical Probability

The empirical probability or the experimental perspective evaluates probability through thought experiments. For example, if a weighted die is rolled, such that we don't know which side has the weight, then we can get an idea for the probability of each outcome by rolling the die number of times and calculating the proportion of times the die gives that outcome and thus find the probability of that outcome.

Subjective Probability

Subjective probability considers an individual's own belief of an event occurring. For example, the probability of a particular team winning a football match on a fan's opinion is more dependent upon their own belief and feeling and not on a formal mathematical calculation.

Axiomatic Probability

In axiomatic probability, a set of rules or axioms by Kolmogorov are applied to all the types. The chances of occurrence or non-occurrence of any event can be quantified by the applications of these axioms, given as,

- The smallest possible probability is zero, and the largest is one.
- An event that is certain has a probability equal to one.
- Any two mutually exclusive events cannot occur simultaneously, while the union of events says only one of them can occur.

Finding the Probability of an Event

In an experiment, the probability of an event is the possibility of that event occurring. The probability of any event is a value between (and including) "0" and "1".

Events in Probability

In probability theory, an event is a set of outcomes of an experiment or a subset of the sample space.

If P(E) represents the probability of an event E, then, we have,

- $P(E) = 0$ if and only if E is an impossible event.
- $P(E) = 1$ if and only if E is a certain event.
- $0 \leq P(E) \leq 1$.

Suppose, we are given two events, "A" and "B", then the probability of event A, $P(A) > P(B)$ if and only if event "A" is more likely to occur than the event "B". Sample space(S) is the set of all of the possible outcomes of an experiment and n(S) represents the number of outcomes in the sample space.

$P(E) = n(E)/n(S)$

$P(E') = (n(S) - n(E))/n(S) = 1 - (n(E)/n(S))$

E' represents that the event will not occur.

Therefore, now we can also conclude that, $P(E) + P(E') = 1$

Coin Toss Probability

Let us now look into the probability of tossing a coin. Quite often in games like cricket, for making a decision as to who would bowl or bat first, we sometimes use the tossing of a coin and decide based on the outcome of the toss. Let us check as to how we can use the concept of probability in the tossing of a single coin. Further, we shall also look into the tossing of two and three coming respectively.

Tossing a Coin

A single coin on tossing has two outcomes, a head, and a tail. The concept of probability which is the ratio of favorable outcomes to the total number of outcomes can be used to find the probability of getting the head and the probability of getting a tail.

Total number of possible outcomes = 2; Sample Space = {H, T}; H: Head, T: Tail

- P(H) = Number of heads/Total outcomes = 1/2
- P(T)= Number of Tails/ Total outcomes = 1/2

Tossing Two Coins

In the process of tossing two coins, we have a total of four outcomes. The probability formula can be used to find the probability of two heads, one head, no head, and a similar probability can be calculated for the number of tails. The probability calculations for the two heads are as follows.

Total number of outcomes = 4; Sample Space = {(H, H), (H, T), (T, H), (T, T)}

- P(2H) = P(0 T) = Number of outcome with two heads/Total Outcomes = 1/4
- P(1H) = P(1T) = Number of outcomes with only one head/Total Outcomes = 2/4 = 1/2
- P(0H) = (2T) = Number of outcome with two heads/Total Outcomes = 1/4

Tossing Three Coins

The number of total outcomes on tossing three coins simultaneously is equal to $2^3 = 8$. For these outcomes, we can find the probability of getting one head, two heads, three heads, and no head. A similar probability can also be calculated for the number of tails.

Total number of outcomes = $2^3 = 8$ Sample Space = {(H, H, H), (H, H, T), (H, T, H), (T, H, H), (T, T, H), (T, H, T), (H, T, T), (T, T, T)}

- P(0H) = P(3T) = Number of outcomes with no heads/Total Outcomes = 1/8
- P(1H) = P(2T) = Number of Outcomes with one head/Total Outcomes = 3/8
- P(2H) = P(1T) = Number of outcomes with two heads /Total Outcomes = 3/8
- P(3H) = P(0T) = Number of outcomes with three heads/Total Outcomes = 1/8

Dice Roll Probability

Many games use dice to decide the moves of players across the games. A dice has six possible outcomes and the outcomes of a dice is a game of chance and can be obtained by using the concepts of probability. Some games also use two dice, and there are numerous probabilities that can be calculated for outcomes using two dice. Let us now check the outcomes, their probabilities for one dice and two dice respectively.

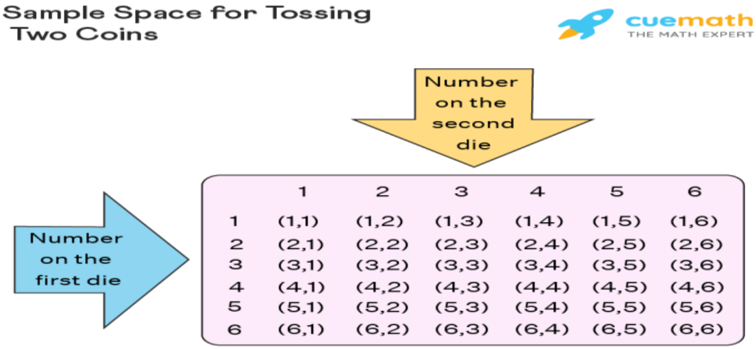
Rolling One Dice

The total number of outcomes on rolling a die is 6, and the sample space is {1, 2, 3, 4, 5, 6}. Here we shall compute the following few probabilities to help in better understanding the concept of probability on rolling one dice.

- P(Even Number) = Number of even number outcomes/Total Outcomes = 3/6 = 1/2
- P(Odd Number) = Number of odd number outcomes/Total Outcomes = 3/6 = 1/2
- P(Prime Number) = Number of prime number outcomes/Total Outcomes = 3/6 = 1/2

Rolling Two Dice

The total number of outcomes on rolling two dice is $6^2 = 36$. The following image shows the sample space of 36 outcomes on rolling two dice.



Sample Space for Tossing Two Coins

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

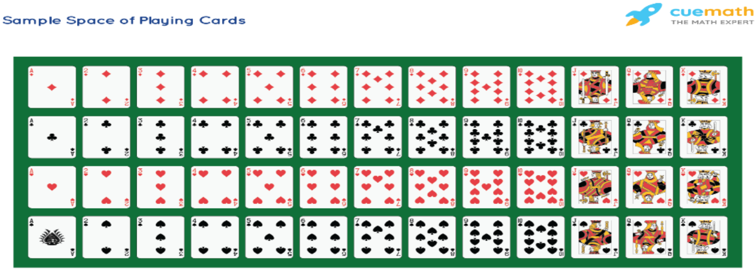
Let us check a few probabilities of the outcomes from two dice. The probabilities are as follows.

- Probability of getting a doublet(Same number) = 6/36 = 1/6
- Probability of getting a number 3 on at least one dice = 11/36
- Probability of getting a sum of 7 = 6/36 = 1/6

As we see, when we roll a single die, there are 6 possibilities. When we roll two dice, there are 36 possibilities. When we roll 3 dice we get 216 possibilities. So a general formula to represent the number of outcomes on rolling 'n' dice is 6^n.

Probability of Drawing Cards

A deck containing 52 cards is grouped into four suits of clubs, diamonds, hearts, and spades. Each of the clubs, diamonds, hearts, and spades have 13 cards each, which sum up to 52. Now let us discuss the probability of drawing cards from a pack. The symbols on the cards are shown below. Spades and clubs are black cards. Hearts and diamonds are red cards.



The 13 cards in each suit are ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king. In these, the jack, the queen, and the king are called face cards. We can understand the card probability from the following examples.

- The probability of drawing a black card is P(Black card) = 26/52 = 1/2
- The probability of drawing a hearts card is P(Hearts) = 13/52 = 1/4
- The probability of drawing a face card is P(Face card) = 12/52 = 3/13
- The probability of drawing a card numbered 4 is P(4) = 4/52 = 1/13
- The probability of drawing a red card numbered 4 is P(4 Red) = 2/52 = 1/26

Probability Theorems

The following theorems of probability are helpful to understand the applications of probability and also perform the numerous calculations involving probability.

Theorem 1: The sum of the probability of happening of an event and not happening of an event is equal to 1. P(A)+P(¯A)=1P(A)+P(A¯)=1

Theorem 2: The probability of an impossible event or the probability of an event not happening is always equal to 0. P(ϕ)=0P(ϕ)=0

Theorem 3: The probability of a sure event is always equal to 1. $P(A) = 1$

Theorem 4: The probability of happening of any event always lies between 0 and 1. $0 \leq P(A) \leq 1$

Theorem 5: If there are two events A and B, we can apply the formula of the union of two sets and we can derive the formula for the probability of happening of event A or event B as follows.

$P(A\cup B)=P(A)+P(B)-P(A\cap B)P(A\cup B)=P(A)+P(B)-P(A\cap B)$

Also for two mutually exclusive events A and B, we have $P(A \cup B) = P(A) + P(B)$

Bayes' Theorem on Conditional Probability

Bayes' theorem describes the probability of an event based on the condition of occurrence of other events. It is also called conditional probability. It helps in calculating the probability of happening of one event based on the condition of happening of another event.

For example, let us assume that there are three bags with each bag containing some blue, green, and yellow balls. What is the probability of picking a yellow ball from the third bag? Since there are blue and green colored balls also, we can arrive at the probability based on these conditions also. Such a probability is called conditional probability.

The formula for Bayes' theorem is $P(A|B)=P(B|A)\cdot P(A)P(B)P(A|B)=P(B|A)\cdot P(A)P(B)$

where, $P(A|B)P(A|B)$ denotes how often event A happens on a condition that B happens.

where, $P(B|A)P(B|A)$ denotes how often event B happens on a condition that A happens.

$P(A)P(A)$ the likelihood of occurrence of event A.

P(B)P(B) the likelihood of occurrence of event B.

Law of Total Probability

If there are n number of events in an experiment, then the sum of the probabilities of those n events is always equal to 1.

P(A1)+P(A2)+P(A3)+....P(An)=1P(A1)+P(A2)+P(A3)+.... P(An)=1

Important Notes on Probability:

Let us check the below points, which help us summarize the key learnings for this topic of probability.

- Probability is a measure of how likely an event is to happen.
- Probability is represented as a fraction and always lies between 0 and 1.
- An event can be defined as a subset of sample space.
- The outcome of throwing a coin is a head or a tail and the outcome of throwing dice is 1, 2, 3, 4, 5, or 6.
- A random experiment cannot predict the exact outcomes but only some probable outcomes.

§14. The concept of quantity. Scalar quantities and properties

The concept of quantity

The concept of quantity. The concept of quantity is one of the basic concepts of mathematics, its formation and formation has a very long history. Along with the development of mathematical science, the meaning of quantitative concepts has also been subjected to a number of generalizations. Even in Euclid's "Begins" the properties of quantities, now called positive scalar quantities, are explained. Quantitative concepts, like other mathematical concepts, have been formed as a result of the needs of people's practical needs. Since ancient times, people have had a serious need to study and compare the properties of various physical, geometric and real objects. Let's look at two of the elements with a straight line piece and a flock of sheep. Just as many groups can be compared, the elements of each set can be compared to each other.

By comparing the elements of each of the clusters we want to consider, let us identify the features common to the elements of these clusters.

By means of a scale liner.

1) Separate even and uneven line sections can be separated from the straight line sections provided;

2) which of the uneven linear lines can be specified as large or small;

3) Each straight line can be divided into several smaller pieces.

By counting.

1) it is possible to distinguish between equal and non-equal flocks of sheep from the flock;

2) it is possible to determine which sheep are more or less than the small number of sheep;

3) Each sheep can be divided into several smaller flocks, with equal numbers of sheep.

From this we can see that although the elements of the clusters we examine differ qualitatively, they have common properties. Many elements are called quantum elements when they have these properties. Since each specific type of quantitative concept is related to a specific comparison of physical, geometric, and other objects, the total quantitative concept is more specific in terms of length, area, volume, mass, and so on. is a direct generalization of quantities.

Definition. The elements of the majority are called quantities, which can be defined by the relationship of equality, magnitude and smallness between the elements, and can divide each element into several equal smaller elements.

For example, the mass of the body is quantitative; because the mass of one body may be larger, smaller or equal to the mass of another. In mathematics, two types of quantities are considered, including discrete and interrupted quantities. Examples of discrete quantities include forest, abundance of natural numbers, and other quantities. Examples of continuous quantities are length (width, height, depth, thickness), surface (area), volume (capacity), mass, time, value, temperature, thermal capacity, specific weight, velocity, angle, linear section and so on. values. We use only unchanged quantities in the issues we will comment on. Therefore, we will shortly use the quantity instead of the quantity.

Quantities:

1) homogeneous quantities;

2) are divided into two groups, including non-uniform (continuous quantity) quantities.

Quantities expressing the same properties of objects of a certain number are called single quantities. Quantities that represent different properties of objects are called non-uniform quantities. For example, length and mass are quantities that are not uniform in size or time. Quantitative numbers are defined as "equal", "big", "small" relationships for quantities. These quantities can be collected and subtracted. To study the quantities in mathematical ways, it is necessary to discover that they have the properties of numbers.

To do this, let us show the following properties of quantities true for numbers:

1. Optional two-dimensional quantities are comparable, that is, two-dimensional quantities are equal or less than one. In other words, relationships of “equal”, “small” and “great” are equally important for the same-sex quantities, and can be true of one and only one of two relations for the numbers a and b.

2. Two identical values can be collected, and the result of the set is the value of the same sex. In other words, for each of the two quantities a and b, the sum of their a+b total is one-dimensional.

3. The value can be shot in real numbers. As a result, the quantity of the same sex is obtained.

4. Two identical values can be deduced, a-b=c and the difference is then determined by the sum, that is, the difference of a and b is such c that $a = b + c$ the relation is correct.

The following basic quantities are studied in mathematics:

1. Length (item length, width, height, thickness, etc.).

2. Area (area of geometric figure, certain geographical area, land area, etc.).

3. Volume (where the body is in space).

4. Mass (amount of substance).

5. Time (the duration of the event or process). Time goes on without interruption, because it is impossible to save or reverse it.

Scalar quantities and properties.

Some of the quantities we encounter in mathematics, physics, mechanics, chemistry, and other sciences are completely determined by the number obtained by measurement. Such quantities are called scalar values. Examples of scalar quantities are length, area, mass, temperature, volume, and so on. can be displayed as values.

Definition. Only quantifiable quantities are called scalar quantities. There are also quantities that need to be specified, not just their numerical values, but their directions. Such quantities are called vectorial quantities or simply vectors.

Definition. Quantities are determined by the numerical value and direction are called vectorial numbers or vectors.

Examples of vectorial quantities are speed, speed, force, and so on. can be displayed as values.

Therefore, the quantities can be divided into two groups:

1. Scalar quantities
2. Vectorial quantities

We only learn positive scalar values.

All scalar quantities have a number of common features. Consider these properties as an example of the length of the piece.

1. Compare any two straight line pieces, such as AB and CD (by length). Based on this comparison, we find that there is one of the following three relationships between these two parts: either AB=CD, either AB<CD either AB>CD.
2. Any two pieces can be collected.
3. Each piece can be divided into smaller equal parts.

If we are talking about the same scalar quantity, such quantities are the same, or if we are talking about different scalar quantities, they are called differential. Since homogeneous quantities are scalar quantities, they have all the properties of scalar quantities. Therefore, homogeneous quantities can be compared, collected, subtracted, multiplied by any real non- negative number, and subdivided into positive real numbers.

GLOSSARY

Volume-həcm
İnteger-tam
Measure-ölçü
Quantity-kəmiyyət
Negative-mənfi
Positive-tam
Multiple-vurma
Number-ədəd
Sum-cəm
Dimension-ölçü
Homogeneous-bircinsli
scalar quantities-skalyar kəmiyyyətlər
real-həqiqi
uneven-qeyri bərabər
counting-saymaq hesablamaq
Time- vaxt
Area-sahə
identical values-eyni qiymət
Mass-kütlə
Length-uzunluq (width- en, height-hündürlük, depth-dərinlik, thickness-qalınlıq), surface-səthi (area-sahə), volume-həcm
Equal-bərabər
Non-equal-qeyri bərabər
mathematical symbols –riyazi simvollar
(capacity-tutum), mass-kütlə, time-vaxt, value-qiymət, temperature-temperatur, thermal capacity-istilik tutumu, specific weight-xüsusi çəkisi, velocity-sürət, angle-bucaq, linear-xətti,
section-hissə
determine-müəyyən etmək
clusters-çoxluqlar
unit-vahid
divide-bölmək

unit of measure-ölçü vahidi
subtract-çıxmaq
multiple-vurmaq
subdivide into-nəyə isə bölmək
continuous quantity-kəsilməz kəmiyyətlər
Fill-doldurun
Complete-tamamlayın
Timetable-cədvəl
Side-tərəf
Metric dimensions-metrik ölçülər
Calculate- hesabla
Area-sahə
Squares-kvadrat
Rectangles-düzbucaqlı
Century-əsr
Weight-çəki, kütlə

EXAMPLES

Example 1. Complete the metric dimensions table:

1 km = ... m = ... dm = ... cm = ... mm
1 m = ... dm = ... cm = ... mm.

Example 2. Fill in smaller sizes:

8km 90mby m; 103m 9dm with desimeters;
76m 7dm 5mm in millimeters;
4 km9m 12cm in centimeters;
140m 12mm in millimeters;
8ha 5m^2 in square meters;
8s 10kg in kilograms;
7t 12s 5kg in kilograms.

Example 3. Convert to large size:

7008 cm in meters;

208902 cm in meters;
1000807 mm in meters;
308050 cm in meters;
54210 mm in centimeters;
108402 kilometers in meters;
82400 kg by centner;
2045060 kg by centner;
3049050 kg in Tons ;
5357 minutes in hours.

Example 4. Complete the weight metric dimensions table:

1T = ... s = ... kq = ... q;
1s = ... kq = ... q.

Example 5. Fill in smaller sizes:

15 t 12 kg in kilograms; 21 kg 70 g In g; 40 kg 8 g of grams; 4 t 40 kg 900 g in g; 8 kg 600 g with g; 4 t 8 kg 80g in g . 8 hours 7 minutes with minutes; 5 hours 6 minutes 7 seconds in sec; 18 km by 25m in m ; 10 t 5 s 18 kg in kilograms.

Example 6. Convert to:

50809 kg to tons; 200800 g in kilograms; 202600 g in kilograms; 1720 seconds to minutes; 502400 m in kilometers; 2448 hours a day.

Example 7. Express in meters:

15 km 18 m, 21 km 102 m, 105 km 28 m, 205 km 112 m.

Example 8. Express in large sizes:

1200 min; 10500 sec; 2025 days; 20,500 coins,
205,450 coins; 204500 centners; 105800 kg.

Example 9. Calculate the following:

1) 5km210m + 6km 890m;
2) 5 hours 15 + 17 hours 87 minutes;

3) $5m^2 10dm^2 + 17m^2 + 27dm^2$;

4) 10km 110m 12dm 5sm + 6km 85dm

5) 1 year 110 days 25 hours + 3 years 302 days 12 hours.

Example 10. Complete the following timetable:

1 century = ... year; 1 simple year = ... days;

1 long year = ... day; 1 year = ... month; 1 year = ... month; 1 week = ... day;

1 day = ... hour; 1 hour = ... minutes;

1 minute = ... seconds; 1 year = ... week.

Example 11. Determine which century belongs years are :

274; 308; 832; 1030; 1348;

1505; 1645; 1954; 2202; 2009; 1953; 2002; 1978.

Example 12. Complete the metric dimensions of the fields:

$1km^2$ = ... hec = ... ar = ... m^2;

1ha = ... a = ... m^2;

1a = ... m^2;

$1m^2$ = ... dm^2 = ... cm^2 = ... mm^2;

$1dm^2$ = ... cm^2 = ... mm^2;

$1sm^2$ = ... mm^2.

Example 13. The sides are 508 m and 175 m; 182 dm and 105 dm; 405 cm and 125 cm; 3 km 200m and 650m; 5 km 25 m and 1 km 250 m. Calculate and then write the area of the rectangles for the numbers of each numerical complex.

Example 14. Sides 206 m; 7 m 9 cm; 380 m; 7 km 80 m; 5 km 40 m; 9 m 50 cm ,Calculate and write the areas of the squares for each of the numbers .

Example 15. Complete the metric dimensions table:

$1m^3$ = ... dm^3 = ... cm^3 = ... mm^3;

$1dm^3$ = ... cm^3 = ... mm^3;

$1sm^3$ = ... mm^3;

$1km^3 = m^3$.

16. Calculate:

a) 102m*5;
b) 2m*12;
c) $205m^2$:5;
d) 1km 50m *8;
e) 8 centners 4 kq :4;
f) 10m 12dm *8;

17. Calculate the volume of the box in the form of a cubic with the side 4m 5 dm and write the result with a number called compound.

18. Calculate the volume of the box in the form of a rectangular paralelepiped with a 4 m 5 dm in length; width of 2 m 4 cm and a height of 3 m 8 cm, and write the result as the compound number

Questions about the last theme

1. What about was mentioned in Euclid's "Begins"?
2. What kind of the basic concepts of mathematics do you know?
3. How mathematical concepts have been formed?
4. Which metods do you know for compare the elements of clusters?
5. What do you explain about the method by the Scale liner?
6. What do you explain about the method by Counting?
7. How many types of quantities includes in math?

8. Examples of discrete quantities include............(complete)

9. Examples of continuous quantities include ....(complete)

10. Definition about scalar quantities....

11. Which basic quantities are studied in mathematics:..

12. Definition about vectorial numbers or vectors.

13. Homogeneous quantities can be... (Complete)

§15. The concept of quantitative measurement

The concept of quantitative. The concept of quantitative measurement.

Measuring any value means comparing that value with another quantity of the same gender and called the unit of measure (for example, meters, kilograms, etc).

The process of comparing quantities depends on the gender of the population in question. Thus, this process is the same for measuring the length, the other for measuring the mass, the other for measuring the volume, and so on. happens. Whatever the process may be, the measured quantity receives a certain numerical value according to the unit of measurement as a result of the measurement.

Definition. The none negative real value of x, which is obtained $a = x \cdot e$ as a result of the measurement of a given unit of quantities by e unit of measurement (a single quantity), is called the numerical value of a by unit of measure e.

We can write the phrase "Numerical value of a real x number with a unit of measure e".

$$x = m_e(a)$$

According to the tariff, each value can be represented by any positive real number in the form of a unit of measure (single unit) accepted for that quantity.

For example,

25 m = 25·1 m, 12 kq =12·1 kq

and so on

Sometimes the quantities given are not directly comparable to one another, or are difficult. In the same way, they are either unable or unwilling to act on them. Measuring quantities allows us to avoid such problems. So, instead of comparing the quantities themselves, we compare the numbers that show their dimensions, and then determine what the relation between the quantities is. Similarly, the quantities do not work on themselves, but on the numbers that indicate their size, and we determine the results of the corresponding actions on the quantities.

This means that quantitative measurements, comparisons of quantities, and performance of quantities allow comparisons of numbers to indicate the same size of units, and the corresponding actions on these numbers.

Suppose that the quantities a and b are expressed in terms of the unit e and the numbers.

$m_e(a)$ and $m_e(b)$ is the relation between the quantities a and b, and the relation between the quantities a and b, as well as the relation between the numbers; Conversely, if there is a relation between the quantities a and b, then there is the same relationship between the numbers and their units.

This statement can be expressed with the help of mathematical symbols as follows:

To find the size of a and b, which is the sum of a and b, the sum of the units of a and b is simply enough to represent the dimensions of $m_e(a)$ and. $m_e(b)$

With the help of mathematical symbols we can write

$$a = b \Leftrightarrow m_e(a) = m_e(b),$$

$$a > b \Leftrightarrow m_e(a) > m_e(b).$$

It is possible to write in the form.

$$a < b \Leftrightarrow m_e(a) < m_e(b),$$

For example, if a = 9 m and b = 14 m,
a + b = 9 m +14 m = (9 +14) m =23 m
can.

Suppose that the relation between the quantities a and b is equal. That is, the value b is equal to the product of a non-negative real x. In this case, in order to find $m_e(b)$ the number that represents the size of b with the unit of measure e, it is sufficient to multiply $m_e(a)$ the integer a by the unit of measure e into a non-negative x.

With the help of mathematical symbols

$$b = xa \Leftrightarrow m_e(b) = x \cdot m_e(a).$$

It is possible to write in the form.

For example, let's say that the volume of one object is b and the volume of the second is a. Also, the size of the first body is six times larger than the second $b = 6a$. If it is $a = 15m^3$, then it is $b = 6a = 6 \cdot (15m^3) = (6 \cdot 15)m^3 = 90m^3$.

GLOSSARY

mathematical symbols-riyazi simvollar
sufficient-kifayətdir
integer-tam
non-negative-mənfi olmayan
unit of measure-ölçü vahidi
same relationship-eyni münasibət
determine-müəyyənləşdirmək
In this case- bu halda

Volume-həcm
Conversely-Əksinə
Suppose-güman etmək,hesab etmək
depends on-asılı olmaq
In the same way-eyni yolla
Whatever-nə olursa olsun
İndicate-göstərmək
can be represented-təmsil oluna bilir
performance of quantities-kəmiyyətlərin icrası
a certain numerical value- müəyyən bir ədədi qiymət
value-qiymət
unit-vahid,hissə
gender -cins
calculate-hesablamaq

QUESTİONS

1. What we have to use for measuring and comparing of the same quantites?
2. How can be writed the phrase-- "Numerical value of a real x number with a unit of measure e". by the formula
3. With the help of mathematical symbols $b = xa$ write this formula on equivalent form
4. If known that the volume of one object is b and the volume of the second is a. Also, the size of the first body is six times larger than the second and as you know, $a = 15m^3$, calculate $b = 6a$

EXAMPLES

1. Calculate and write the areas of the squares for this number : Sides 206 m;

2. Express in large sizes:

1200 min; 10500 sec; 2025 days; 20,500 coins

3. Express in large sizes:

205,450 coins; 204500 centners; 105800 kg.

4. Calculate the volume of the box in the form of a rectangular paralelepiped with a 4 m 5 dm in length; width of 2 m 4 cm and a height of 3 m 8 cm, and write the result as the compound number.

5. Determine which century belongs years are :

274; 308; 832; 1030; 1348;

6. Determine which century belongs years are :

1505; 1645; 1954; 2202; 2009; 1953; 2002; 1978.

7. 2.Calculate the following:

1 year 110 days 25 hours + 3 years 302 days 12 hours.

§16. Length units and their connection

Length units

Relationship between units of length measurement

In ancient times, people used their steps, feet, ants, etc. to measure length. But when people live in a team, they need the same and more accurate measurement units, or they wouldn't understand each other. Therefore, the need for the use of a unit of measure has been established in ancient times.

The idea of creating a new measurement system first appeared in France after the 1789 Revolution. At present, metric measurements are being developed in all countries of the world. Metric size was created in France at the end of the eighteenth century. The makers of this dimension sought to find a unit of length that would have been taken from nature, that is, to match the length of any distance on earth.

For this purpose, they have found the length of the earth's meridian. The length of the straight line, which is 10 million times the quarter of the Earth's meridian, is assumed to be the principal unit of length. This length is called the meter. "Metr" is a Greek word meaning "measure" in Azerbaijani (uzuznluq).

There are also units of length and meters smaller than meters. For the metric units of length, meters are used before the word “meter”, which contains the Greek words deac (ten), hecto (one hundred), and weight (thousand). In order to get the names of units of length less than meters, pre-shaped desi (one in ten), santa (one per cent) and national (one in a thousand) prefabricators are used.

So the table of length units is taken:

1 kilometr (km)=10 hektometr (hm)=1000 m

1 hektometr (hm)=10 dekametr (dkm)=100 m

1 dekametr (dkm)=10 m
1 metr (m)=10 desimetr (dm)=100 santimetr (sm)
1 desimetr (dm) = 10 santimetr (sm)=0,1 m
1 santimetr (sm) = 10 millimetr (mm)=0,01 m
1 geographical miles = 7,420 km
1 miles of sea = 1,852 km

Sometimes the following units of length are also used.

1 meqametr = 1000000 m
1 miriametr (Mm) = 10 000 m
1 millimetr (mm)=1000 mikron (μ)=0,001 m
1 mikron (mikrometr) (μ)=1000 millimikron(mμ)=0,000001 m
1millimikron =0,000000001 m

Microns and millimeters are used only in studies performed with the help of a microscope.

Measurement of area. Area measurement units.

History

Archimedes (287–212 B.C.E.) is considered to have been the greatest mathematician in antiquity. Method for deriving the volume of a sphere, called the Archimedean method, involved a lever principle. He compared a sphere of radius r and a cone of radius 2r and height 2r to a cylinder also of radius 2r and height 2r. Using cross-sections, Archimedes deduced that the cone and sphere as solids, placed two units from the fulcrum of the lever, would balance the solid cylinder placed one unit from the fulcrum.

In fact, he is ranked by many with Sir Isaac Newton and Carl Friedrich Gauss as one of the three greatest mathematicians of all time. Perhaps the most famous story about him is that of his discovery of the principle of buoyancy; namely, a body immersed in water is buoyed up by a force equal to the weight of the water displaced. Legend has it that he discovered the buoyancy principle while bathing and was so excited that he ran naked into the street shouting “Eureka!”

In mathematics, Archimedes discovered and verified formulas for the surface area and volume of a sphere. Hence, the volume of the cone plus the volume of the sphere equals the volume of the cylinder. However, the volume of the cone was known to be the volume of the cylinder, so that the volume of the sphere must be the volume of the cylinder. Thus the volume of the sphere is , which is . The original description of the Archimedean method was thought to be permanently lost until its rediscovery in Constantinople, now Istanbul, in 1906. Archimedes is credited with anticipating the development of some of the ideas of calculus, nearly 2000 years before its creation by Sir Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716).

Use Dimensional Analysis

The strategy Use Dimensional Analysis is useful in applied problems that involve conversions among measurement units. For example, distance–rate– time problems or problems involving several rates (ratios) are sometimes easier to analyze via dimensional analysis. Additionally, dimensional analysis allows us to check whether we have reported our answer in the correct measurement units.

The assumptions that people have about the area are much broader. Because people are more involved with this concept in practice. Concerning the area of apartments, the land they own, the area of the walls that is required to be painted, it is often talked about. Thus, since the concept of a field is more closely related to human beings, it is often referred to as measuring, comparing, and other such issues. Therefore, people are well aware that if two pieces of land are equal, their area is equal, the area of a large piece of land is larger, and that the area of the apartment is composed of separate rooms.

Initial problem

David was planning a motorcycle trip across Canada. He drew his route on a map and estimated the length of his route to be 115 centimeters. The scale on his map is 1 centimeter _ 39 kilometers. His motorcycle's gasoline consumption averages 75 miles per gallon of gasoline. If gasoline costs $3 per gallon, how much should he plan to spend for gasoline? (Hint: 1 mile is approximately 1.61 kilometers.)

Glues

The Use Dimensional Analysis strategy may be appropriate when

- Units of measure are involved.
- The problem involves physical quantities.
- Conversions are required.

Domestic considerations about the area are widely used when finding areas of geometric figures. Because geometric figures have different structures, they separate specific classroom figures when talking about their areas. For example, areas of figure classes, such as polygons and limited void shapes, circles and other rotation objects, are studied separately. Only the areas of polygons and other limited edges are considered here.

It can be composed of optional F figure figures. The fact that the figure F is composed of a number of figures means that the F figure is a combination of constituent figures and does not have a common element of two separate figures.

For example, if the figure F is composed of figures, the F figure is a combination of the constituent figures and does not have a single element of the two constituent figures (Figure 3).

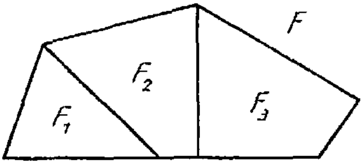


Figure 3.

Definition. The area of the figure is called a non-negative quantity that pays the following conditions for each figure:

1) The areas of equal figures are also equal.

2) If the figure is made up of a finite number of units, then its area is the sum of the areas of these figures.

When comparing this definition with the length of the piece, it is clear that the area is also characterized by the properties of the length of the piece. However, field and length quantities are assigned to different clusters (with lots of length cloths, and field numbers with independent figures).

Definition. The figure with all its points on a single plane is called a regular figure.

Let us denote the area of the figure F as S (F). We illustrated the essence of the measurement process as an example of measuring the length of the pieces. The process of site measurement is also carried out using the same scheme. Thus, the prerequisite for measuring areas is to select (accept) a unit of measurement. The unit of area is the square with the side equal to any unit of length. Such a square is called a square unit and is taken as a unit of area. There is a square of the square with the side e. If the side of such a square is 1 centimeters, this square is called square, and if its side is 1 meter, then it is called square meters, and so on.

Measuring the area is a process of comparing the square of a given figure with a single square. The result of such a measurement is a positive number.

Consequently, the value of the figure area is a positive number indicating how many times the unit area (single square) is located in the given figure.

Definition. The positive real number m|n that satisfies $S(F)=\frac{m}{n}\cdot e^2$ **the condition is called the area of the square F circle with square units** (e^2)**.**

The definition of a field is derived from a number of properties that allow it to establish rules for comparing areas and performing actions on fields. Let's look at some of these.

Property 1. If the figures given are equal, then the numbers with the same area unit are the same.

The property is not the opposite. Thus, if the areas of the figures are equal, they may not be equal.

Definition. Figures with equal areas are called as same figures .

Note Equal figures are always one size, but in one size the figures may not be equal.

Property 2. If the figure F is made of figures F1, F2 and F3, then the numbers representing the area of any area

of the figure F are the sum of the numbers representing the areas of the F1, F2, and F3 figures.

S (F) =S (F1) + S (F2) + S (F3)

Property 3. When replacing a unit unit with a new unit of unit n times smaller than itself, the number of units showing the area of the unit and its area increases (decreases).

To calculate the area of a rectangle, it is necessary to measure its length and width by the same unit of measure, and then subtract the numbers. The extracted rectangle will show how many square units are in the area.

For example, if the rectangle is 5 cm in length and 3 cm in width, the area is 5*3=15 (sm^2)

In the metric system, the upper square unit is 100 times larger than the lower square unit.

Area unit schedule.

1 km^2 = 100 ha = 1000 000 m^2

1 ha =100 ar = 10 000 m^2

1 ar = 100 m^2

1 m^2 = 100 dm^2 = 10 000 sm^2

1 dm^2 = 100 sm^2 = 10 000 mm^2

1 sm^2 = 100 mm^2

For example, let's express 7m ^{2} with square desimeter. The square desimeter is 100 times smaller than the square meter, mean that

$$1m^2 = 100dm^2$$

because of that

$$7m^2 = 7 \cdot 1\ m^2 = 7 \cdot (100dm^2) = \\ = (7 \cdot 100)\, dm^2 = 700dm^2$$
.

• **PREKINDERGARTEN**: Identifying measurable attributes and comparing objects by using these attributes.

• **GRADE 1**: Composing and decomposing geometric shapes.

• **GRADE 2:** Developing an understanding of linear measurement and facility in measuring lengths.

• **GRADE 4**: Developing an understanding of area and determining the areas of two-dimensional shapes.

• **GRADE 5:** Describing three-dimensional shapes and analyzing their properties, including volume and surface area.

• **GRADE 7:** Developing an understanding of and using formulas to determine surface areas and volumes of three-dimensional shapes. he measurement process allows us to analyze geometric figures using real numbers. For example, suppose that we use a sphere to model the Earth. Then we can ask many questions about the sphere, such as “How far is it around the equator? How much surface area does it have? How muchspace does it take up?” Questions such as these can lead us to the study of the measurement of length, area, and volume of geometric figures, as well as other attributes.In the first section of this chapter we introduce **holistic measurement**, using natural or **nonstandard units** such as “hands” and “paces.” We also study two systems of standard units, namely the English system or customary system of units, which we Americans use, and the metric system, or Systѐme International (SI), which virtually all other countries use. In the other sections, we study abstract mathematical measurement of geometric shapes, exploring length, area, surface area, and volume.

Nonstandard Units

The measurement process is defined as follows. When carpet is purchased for a room, a salesman might ask “how many yards do you need?” When ordering concrete to pour a sidewalk, the dispatcher will ask “how many yards do you need?” Are yards in both of these situations the same? Explain.

The Measurement Process

1. Select an object and an attribute of the object to measure, such as its length,

area, volume, weight, or temperature.

2. Select an appropriate unit with which to measure the attribute.

3. Determine the number of units needed to measure the attribute. (This may require

a measurement device.)

DEFINITION

As students use nonstandard,and even invented, units ofmeasurement, they are able togain a deeper understanding ofmeasurement and some of itsproperties. It also enhances theirunderstanding of standard

measurement (Young &O’Leary, 2002). For example, to measure the length of an object, we might see how many times our hand will span the object. Figure 13.2 shows a stick that is four “hand spans” long“Hands” are still used as a unit to measure the height of horses.

For measuring longer distances, we might use the length of our feet placed heel to toe or our pace as our unit of measurement. For shorter distances, we might use the width of a finger as our unit (Figure 13.3). Regardless, in every case, we can select some appropriate unit and determine how many units are needed to span the object. This is an informal measurement method of measuring length, since it involves naturally occurring units and is done in a relatively imprecise way. 1 hand A horse that is 16 hands tall Stepping off

a room A flower that is 5 fingers wide. To measure the area of a region informally, we select a convenient two dimensional shape as our unit and determine how many such units are needed to cover the region. Figure 13.4 shows how to measure the area of a rectangular rug, using square floor tiles as the unit of measure. By counting the number of squares inside the rectangular border, and estimating the fractional parts of the other squares that are partly inside the border, it appears that the area of the rug is between 15 and 16 square units (certainly, between 12 and 20).

NCTM Standard

All students should understand how to measure using nonstandard and standard units.

Reflection from Research

A possible teaching sequence for area is to “investigate rectangular covering as a problem in itself, then introduce the area concept” and then use knowledge of covering to measure area (Outhred & Mitchelmore, 2000). To measure the capacity of water that a vase will hold, we can select a convenient container, such as a water glass, to use as our unit and count how many glassfuls are required to fill the vase. This is an informal method of measuring volume. (Strictly speaking, we are measuring the capacity of the vase, namely the amount that it will hold. The volume of the vase would be the amount of material comprising the vase itself.) Other holistic volume measures are found in recipes: a “dash” of hot sauce, a “pinch” of salt, or a “few shakes” of a spice, for example. Measurement using nonstandard units is adequate for many needs, particularly when accuracy is not essential. However, there are many other circumstances when we need to determine measurements more precisely and communicate them to others. That is, we need standard measurement units as discussed next.

Standard Units

The English System The **English system** of units arose from natural, nonstandard units. For example, the foot was literally the length of a human foot and the yard was the distance from the tip of the nose to the end of an outstretched arm (useful in measuring cloth or “yard goods”). The inch was the length of three barley corns, the fathom was the length of a full arm span (for measuring rope), and the acre was the amount of land that a horse could plow in one day . Measurement with Nonstandard and Standard Units 669

1 foot 1 inch

1 acre

1 fathom

1 yard

Length The natural English units were standardized so that the foot was defined by a prototype metal bar, and the inch defined as of a foot, the yard the length of 3 feet, and so on for other lengths. A variety of ratios occur among the English units of length. For example, the ratio of inches to feet is 12_1, of feet to yards is 3_1, 1 12

NCTM Standard

All students should develop common referents for measures to make comparisons and estimates.

FRACTION OR MULTIPLE

UNIT OF 1 FOOT

Inch ft

Foot 1 ft

Yard 3 ft

Rod ft

Furlong 660 ft

Mile 5280 ft

1612

1

12

of yards to rods is and of furlongs to miles is 8_1. A considerable amount of memorization is needed in learning the English system of measurement.

Area

Area is measured in the English system using the square foot (written ft2) as the fundamental unit. That is, to measure the area of a region, the number of squares, 1 foot on a side, that are needed to cover the region is determined. This is an application of tessellating the plane with squares . Other polygons could, in fact, be used as fundamental units of area. For example, a right triangle, an equilateral triangle, or a regular hexagon could also be used as a fundamental unit of area. For large regions, square yards are used to measure areas, and for very large regions, acres and square miles are used to measure areas. Table 13.2 gives the relationships among various English system units of area. Here again, the ratios between area units are not uniform.

5 1

2 : 1,

MULTIPLE OF 1 UNIT SQUARE FOOT

Square inch 1/144 ft2

Square foot 1 ft2

Square yard 9 ft2

Acre 43,560 ft2

Square mile 27,878,400 ft2

A more general strategy, called dimensional analysis, appears later in this section. Compute the ratios square feet_square yards and square feet_square miles.

SOLUTION

Since there are 3 feet in 1 yard and 1 square yard measures 1 yard by 1 yard, we see that there are 9 square feet in 1 square yard. Therefore, the ratio of square feet to square yards is 9_1.

Example

3 ft

3 ft

1 yd2

(a) (b)

5280 ft

5280 ft

1 mile2

(not to scale)

Next imagine covering a square, 1 mile on each side, with square tiles, each 1 foot

on a side. It would take an array of square with 5280 rows, each row having 5280 tiles. Hence it would take 5280 _ 5280 _ 27,878,400 square feet to cover 1 square mile. So the ratio of square feet to square miles is 27,878,400_1.

Volume

In the English system, volume is measured using the cubic foot as the fundamental

unit . To find the volume of a cubical box that is 3 feet on each side, imagine stacking as many cubic feet inside the box as possible. The box could be filled with 3 _ 3 _ 3 _ 27 cubes, each measuring 1 foot on an edge. Each of the smaller cubes has a volume of 1 cubic foot (written ft3), so that the larger cube has a volume of 27 ft3. The larger cube is, of course, 1 cubic yard (1 yd3). It is common for topsoil and concrete to be sold by the cubic yard, for example. In the English system, we have several cubic units used for measuring volume.Note the variety of volume ratios in the English system.

Measurement with Nonstandard and Standard Units

FRACTION OR MULTIPLE UNIT OF A CUBIC FOOT

Cubic inch (1 in3) 1/1728 ft3

Cubic foot 1 ft3

Cubic yard (1 yd3) 27 ft3

TABLE 13.3

Verify the ratio of in3_ft3 given in Table 13.3.

SOLUTION Since there are 12 inches in each foot, we

could fill a cubic foot with 12 _ 12 _ 12 smaller cubes, each 1 inch on an edge.

Hence there are 123 _ 1728 cubic inches in 1 cubic foot. Consequently, each cubic

inch is of a cubic foot. ■

UNIT ABBREVIATION RELATION TO PRECEDING UNIT

1 teaspoon tsp

1 tablespoon tbsp 3 teaspoons
1 liquid ounce oz 2 tablespoons
1 cup c 8 liquid ounces
1 pint pt 2 cups
1 quart qt 2 pints
1 gallon gal 4 quarts
1 barrel bar 31.5 gallons

Weight

In the English system, weight is measured in pounds and ounces. In fact, there are two types of measures of weight—troy ounces and pounds (mainly for precious metals), and avoirdupois ounces and pounds, the latter being more common.

We will use the avoirdupois units. The weight of 2000 pounds is 1 English ton. Smaller weights are measured in drams and grains. Table 13.5 summarizes these English system units of weight. Notice how inconsistent the ratios are between consecutive units.

UNIT RELATION TO PRECEDING UNIT

1 grain
1 dram grains
1 ounce 16 drams
1 pound 16 ounces
1 ton 2000 pounds

English System Units of Weight (Avoirdupois)

Technically, the concepts of weight and mass are different. Informally, mass is the measure of the amount of matter of an object and weight is a measure of the force with which gravity attracts the object. Thus, although your mass is the same on Earth and on the Moon, you weigh more on Earth because the attraction of gravity is greater on Earth. We will not make a distinction between weight and mass. We will use English units of weight and metric units of mass, both of which are used to weigh objects.

Temperature

Temperature is measured in **degrees Fahrenheit** in the English system. The Fahrenheit temperature scale is named for Gabriel Fahrenheit, a German instrument maker, who invented the mercury thermometer in 1714. The freezing point and boiling point of water are used as reference temperatures. The freezing point is arbitrarily defined to be 32_ Fahrenheit, and the boiling point 212_ Fahrenheit. This gives an interval of exactly 180_ from freezing to boiling.

The Metric System

In contrast to the English system of measurement units, the **metric system** of units (or Système International d'Unités) incorporates all of the following features of an ideal system of units.

An Ideal System of Units

1. The fundamental unit can be accurately reproduced without reference to a

prototype. (Portability)

2. There are simple (e.g., decimal) ratios among units of the same type.

(Convertibility)

3. Different types of units (e.g., those for length, area, and volume) are defined in

terms of each other, using simple relationships. (Interrelatedness)

Boiling point of water

Body temperature

Room temperature
Freezing point of water
70°F
32°F
98.6°
212°F

NCTM Standard

All students should understand both metric and customary systems of measurements.

Length

In the metric system, the fundamental unit of length is the **meter** (about inches). The meter was originally defined to be one ten-millionth of the distance from the equator to the North Pole along the Greenwich-through-Paris meridian.

A prototype platinum-iridium bar representing a meter was maintained in the International Bureau of Weights and Measures in France. However, as science advanced, this definition was changed so that the meter could be reproduced anywhere in the world. Since 1960, the meter has been defined to be precisely

1,650,763.73 wavelengths of orange-red light in the spectrum of the element krypton 86. Although this definition may seem highly technical, it has the advantage of being reproducible in a laboratory anywhere. That is, no standard meter prototype need be kept. This is a clear advantage over older versions of the English system. We shall see that there are many more. The metric system is a decimal system of measurement in which multiples and fractions of the fundamental unit correspond to powers of ten. For example, one thousand meters is a **kilometer**, one-tenth of a meter is a

decimeter, one-hundredth of a meter is a **centimeter**, and one-thousandth of a meter is a **millimeter**.

Notice the simple ratios among units of length in the metric system. (Compare Table 13.6 to Table 13.1 for the English system, for example.) From Table 13.6 we see that 1 **dekameter** is equivalent to 10 meters, 1 **hectometer** is equivalent to 100 meters, and so on. Also, 1 dekameter is equivalent to 100 decimeters, 1 **kilometer** is equivalent to 1,000,000 millimeters, and so on.

Reflection from Research Students have difficulty estimating lengths of segments using metric units. In a national assessment only 30% of the thirteen-year-olds successfully estimated segment length to the nearest centimeter (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981).

FRACTION OR MULTIPLE UNIT SYMBOL OF 1 METER

1 millimeter 1 mm 0.001 m
1 centimeter 1 cm 0.01 m
1 decimeter 1 dm 0.1 m
1 meter 1 m 1 m
1 dekameter 1 dam 10 m
1 hectometer 1 hm 100 m
1 kilometer 1 km 1000 m

From the information in Table 13.6, we can make a metric "converter" diagram to simplify changing units of length. Locate consecutive metric abbreviations for units of length starting with the largest prefix on the left . To convert from, say, hectometers to centimeters, count spaces from "hm" to "cm" in the diagram, and move the decimal point in the same direction as many spaces as are indicated in the diagram (here, four to the right). For example, 13.23685 hm _ 132,368.5 cm. Similarly, 4326.9 mm _ 4.3269 m, since we move three spaces to the left in the diagram when going from "mm" to "m." It is good practice to use measurement sense as a check. For example, when converting from mm to hm, we have fewer "hm's"

than "mm's," since 1 hm is longer than 1 mm. km hm dam m dm cm mm

The three most commonly used prefixes are in italics. We will see that these prefixes are also used with measures of area, volume, and weight. Notice how the prefixes signify the ratios to the fundamental unit.

1 meter

1 yard

1 foot

1 decimeter

1 inch

1 centimeter

Lengths that are measured in feet or yards in the English system are commonly measuredin meters in the metric system. Lengths measured in inches in the English system are measured in centimeters in the metric system. (By definition, 1 in. is exactly 2.54 cm.) For example, in metric countries, track and field events use meters instead of yards for the lengths of races. Snowfall is measured in centimeters in metric countries, not inches.

Area

In the metric system, the fundamental unit of area is the square meter. A square that is 1 meter long on each side has an area of **1 square meter**, written 1 m^2. Areas measured in square feet or square yards in the English system are measured in square meters in the metric system. For example, carpeting would be measured in square meters. Smaller areas are measured in square centimeters. A **square centimeter** is the area of a square that is 1centimeter long on each side. For example, the area of a piece of notebook paper or a photograph would be measured in square centimeters (cm^2). Determine the number of square centimeters in 1 square meter.

SOLUTION

A square with area 1 square meter can be covered with an array of square centimeters. In Figure 13.14 we see part of the array.

1 m2

1 m

1 m

100 cm

1 m2 100 cm

(not to scale)

There are 100 rows, each row having 100 square centimeters. Hence there are 100 _ 100 _ 10 000 square centimeters needed to cover the square meter. Thus 1 m^2 _ 10 000 cm^2. Very small areas, such as on a microscope slide, are measured using square millimeters. A **square millimeter** is the area of a square whose sides are each 1 millimeter long. In the metric system, the area of a square that is 10 m on each side is given the special name **are** (pronounced “air”). Figure 13.15 illustrates this definition. An are is approximately the area of the floor of a large two-car garage and is a convenient unit for measuring the area of building lots. There are 100 m2 in 1 are. An area equivalent to 100 ares is called a **hectare**, written 1 ha.

Notice the use of the prefix “hect” (meaning 100). The hectare is useful for measuring areas of farms

10 m

1 are 10 m

(100 m^2)

and ranches. We can show that 1 hectare is 1 square hectometer by converting each to square meters, as follows. Also, Thus 1 ha _ 1 hm^2. Finally, very large areas are measured in the metric system using square kilometers. One **square kilometer** is the area of a square that is 1 kilometer on each side. Areas of cities or states, for example, are reported in square kilometers. Table 13.8 gives the ratios among various units of area in the metric system. See if you can verify the entries in the table.

1 hm2 = (100 m) * (100 m) = 10 000 m2.

1 ha = 100 ares = 100 * (100 m2) = 10 000 m2

FRACTION OR MULTIPLE UNIT ABBREVIATION OF 1 SQUARE METER

Square millimeter mm2 0.000001 m2

Square centimeter cm2 0.0001 m2

Square decimeter dm2 0.01 m2

Square meter m2 1 m2

Are (square dekameter) a (dam2) 100 m2

Hectare (square hectometer) ha (hm2) 10 000 m2

Square kilometer km2 1 000 000 m2

Volume

The fundamental unit of volume in the metric system is the liter. A **liter**, abbreviated L, is the volume of a cube that measures 10 cm on each edge. We can also say that a liter is 1 **cubic decimeter**, since the cube in measures 1 dm on each edge.

Notice that the liter is defined with reference to the meter, which is the fundamental unit of length. The liter is slightly larger than a quart. Many soft-drink containers have capacities of 1 or 2 liters.

We see that the metric prefixes for square units should not be interpreted in the abbreviated forms as having the same meanings as with linear units. For example, 1 dm2 is not one-tenth of 1 m2; rather, 1 dm2 is one-hundredth of 1 m2. Conversions among units of area can be done if we use the metric converter in but move the decimal point twice the number of spaces that we move between units. This is due to the fact that area involves two dimensions. For example, suppose that we wish to convert 3.7 m2 to mm2. We move three spaces to the right from “m” to “mm,” so we will move the decimal point 3 _ 2 _ 6 (the “2” is due to the two dimensions) places to the right. Thus 3.7 m2 _ 3 700 000 mm2. Since 1 m2 _ (1000 mm)2 _ 1 000 000 mm2, we have 3.7 m2 _ 3.7 _ (1 000 000) mm2 _ 3 700 000 mm2, which is the same result that we obtained using the metric converter.

NCTM Standard

All students should carry out simple unit conversions such as centimeters to meters, within a system of measurement.

1 L^3 =100 cm^3

Imagine filling the liter cube in with smaller cubes, 1 centimeter on each edge. Each small cube has a volume of 1 **cubic centimeter** (1 cm^3). It will take a 10 _ 10 array (hence 100) of the centimeter cubes to cover the bottom of the liter cube. Finally, it takes 10 layers, each with 100 centimeter cubes, to fill the liter cube to the top. Thus 1 liter is equivalent to 1000 cm^3. Recall that the prefix “milli-” in the metric system means one-thousandth. Thus we see that 1 **milliliter** is equivalent to 1 cubic centimeter, since there are 1000 cm^3 in 1 liter. Small volumes in the metric system are measured in milliliters (cubic centimeters). Containers of liquid are frequently labeled in milliliters.

NCTM Standard

All students should understand such attributes as length, area, weight, volume, and size of angle and select the appropriate type of unit for measuring each attribute.

10 cm = 1 dm
10 cm = 1 mm
10 m = 10 dm

Large volumes in the metric system are measured using cubic meters. A **cubic meter** is the volume of a cube that measures 1 meter on each edge . Capacities of large containers such as water tanks, reservoirs, or swimming pools are measured using cubic meters. A cubic meter is also called a **kiloliter**.

FRACTION OR MULTIPLE UNIT ABBREVIATION OF 1 LITER

Milliliter (cubic centimeter) mL (cm3) 0.001 L
Liter (cubic decimeter) L (dm3) 1 L
Kiloliter (cubic meter) kL (m3) 1000 L

In the metric system, capacity is usually recorded in liters, milliliters, and so on. We can make conversions among metric volume units using the metric converter.

$1\ m^3=1\ m*1\ m*1\ m$

To convert among volume units, we count the number of spaces that we move left or right in going from one unit to another. Then we move the decimal point exactly three times that number of places, since volume involves three dimensions.

Mass

In the metric system, a basic unit of mass is the kilogram. One **kilogram** is the mass of 1 liter of water in its densest state. (Water expands and contracts somewhat\ when heated or cooled.) A kilogram is about 2.2 pounds in the English system. Notice that the kilogram is defined with reference to the liter, which in turn was defined relative to the meter. This illustrates the interrelatedness of the metric units meter, liter, and kilogram. We can conclude that 1 milliliter of water weighs of a kilogram. This weight is called a **gram**. Grams are used for small weights in the metric system, such as ingredients in recipes or nutritional contents of various foods. Many foods are packaged and labeled by grams. About 28 grams are equivalent to 1 ounce in the English system. We can summarize the information in Table 13.9 with the definitions of the various metric weights in the following way. In the metric system, there are three basic cubes: the cubic centimeter, the cubic decimeter, and the cubic meter.

CUBE VOLUME MASS (WATER)

1 m^3 1 kL 1 tonne
1 dm^3 1 L 1 kg
1 cm^3 1 mL 1 g
1 cm^3= 1 mL of water with
mass 1 g
1 dm^3= 1000 cm^3
= 1 L of water
with mass 1 kg
1 m^3= 1000 dm^3= 1,000,000 cm^3= 1 kL of water
with mass 1 tonne

The cubic centimeter is equivalent in volume to 1 milliliter, and, if water, it weighs 1 gram. Similarly, 1 cubic decimeter of volume is 1 liter and, if water, weighs 1 kilogram. Finally, 1 cubic meter of volume is 1 kiloliter and, if water, weighs 1000 kilograms, called a **metric ton** (tonne). **Temperature** In the metric system, temperature is measured in **degrees Celsius**. The Celsius scale is named after the Swedish astronomer Anders Celsius, who devised it in 1742. This scale was originally called “centigrade.” Two reference temperatures are used, the freezing point of water and the boiling point of water. These are defined to be, respectively, zero degrees Celsius (0_C) and 100 degrees Celsius (100_C). A metric thermometer is made by dividing the interval from freezing to boiling into 100 degrees Celsius.

Boiling point of water-100°C
Body temperature-37° C
Room temperature-20°
Freezing point-0°C

The relationship between degrees Celsius and degrees Fahrenheit (used in the English system) is derived next.

a. Derive a conversion formula for degrees Celsius to degrees Fahrenheit.

b. Convert 37_C to degrees Fahrenheit.

c. Convert 68_F to degrees Celsius.

SOLUTION

a. Suppose that C represents a Celsius temperature and F the equivalent Fahrenheit temperature. Since there are 100_ Celsius for each 180_ Fahrenheit , there is 1_ Celsius for each 1.8_ Fahrenheit. If C is a temperature above freezing, then the equivalent Fahrenheit temperature, F, is 1.8C degrees Fahrenheit above 32_ Fahrenheit, or 1.8C _ 32. Thus 1.8C _ 32 _ F is the desired formula. (This also applies to temperatures at freezing or below, hence to all temperatures.)

b. Using 1.8C _ 32 _ F, we have 1.8(37) _ 32 _ 98.6_ Fahrenheit, which is normal human body temperature.

c. Using 1.8C _ 32 _ F and solving C, we find Hence room temperature of 68_ Fahrenheit is equivalent to Celsius. ■

Water is densest at 4_C. Therefore, the precise definition of the kilogram is the mass of 1 liter of water at 4_C.

From the preceding discussion, we see that the metric system has all of the features of an ideal system of units: portability, convertibility, and interrelatedness. These features make learning the metric system simpler than learning the English system of units. The metric system is the preferred system in science and commerce throughout the world. Moreover, only a handful of countries use a system other than the metric system.

C =68 - 32

1.8 = 20°

C =F - 32

1.8.

Example

Reflection from Research

It is difficult for students to comprehend that it takes more inches than feet to cover the same distance. The inverse relationship that exists because inches are smaller but more of them are required than feet can be confusing (Hart, 1984).

Dimensional Analysis

When working with two (or more) systems of measurement, there are many circumstances requiring conversions among units. The procedure known as dimensional analysis can help simplify the conversion. In **dimensional analysis**, we use unit ratios that are equivalent to 1 and treat these ratios as fractions. For example, suppose that we wish to convert 17 feet to inches. We use the unit ratio 12 in./1 ft (which is 1) to perform the conversion. Hence, a length of 17 ft is the same as 204 inches. Dimensional analysis is especially useful if several conversions must be made.

Example.

A vase holds 4286 grams of water. What is its capacity in liters?

SOLUTION

Since 1 mL of water weighs 1 g and 1 L _ 1000 mL, we have Consequently, the capacity of the vase is 4.286 liters. ■

Example

Notice that treating the ratios as fractions allows us to use multiplication of fractions. Thus we can be sure that our answer has the proper units. The area of a rectangular lot is 25,375 ft2. What is the area of the lot in acres? Use the fact that 640 acres _ 1 square mile.

SOLUTION

We wish to convert from square feet to acres. Since 1 mile _ 5280 ft, we can convert from square feet to square miles. That is, 1 mile2 _ 5280 ft _ 5280 ft _ 27,878,400 ft2.

Example

Shows how to make conversions between English and metric system units. We do not advocate memorizing such conversion ratios, since rough approximations serve in most circumstances. However, there are occasions when accuracy is needed. In fact, the English system units are now legally defined in terms of metric system units. Recall that the basic conversion ratio for lengths is 1 inch_2.54 centimeters, exactly.

25,375 * 640=27,878,400acres = 0.58 acre (to two places).

25,375 ft 2 = 25,375 ft2 *

1 mile^2=27,878,400 ft^2 *

NCTM Standard

All students should compare and order objects by attributes of length, volume, weight, area, and time.

Problem Solving Strategy

1. Use Dimensional Analysis A pole vaulter vaulted 19 ft in. Find the height in meters.

SOLUTION

Since 1 meter is a little longer than 1 yard and the vault is about 6 yards, we estimate the vault to be 6 meters. Our final example illustrates how we can make conversions involving different types of units, here distance and time.

2. Suppose that a bullet train is traveling 200 mph. How many feet per second is it traveling?

EXERCISES

1. In your elementary classroom, you find the following objects. For each object, list attributes that could be measured and how you could measure them.

a. A student’s chair **b.** A wastebasket

c. A bulletin board **d.** An aquarium

2. Measure the length, width, and height of this textbook using the following items.

a. Paper clip

b. Stick of gum

c.Will your results for parts a and b be the same as your classmates? Explain.

Problem Solving Strategy

Use Dimensional Analysis

3. Calculate the following.

a. How many inches in a mile?

b. How many yards in a mile?

c. How many square yards in 43,560 square feet?

d. How many cubic inches in a cubic yard?

e. How many ounces in 500 pounds?

f. How may cups in a quart?

g. How many cups in a gallon?

h. How many tablespoons in a cup?

4. Choose the most realistic measures of the following objects.

a. The length of a small paper clip: 28 mm, 28 cm, or 28 m?

b. The height of a 12-year-old boy: 48 mm, 148 cm, or 48 m?

c. The length of a shoe: 27 mm, 27 cm, or 27 m?

5. Choose the most realistic measures of the volume of the following objects.

a. A juice container: 900 mL, 900 cL, or 900 L?

b. A tablespoon: 15 mL, 15 cL, or 15 L?

c. A pop bottle: 473 mL, 473 cL, or 473 L?

6. Choose the most realistic measures of the mass of the following objects.

a. A 6-year-old boy: 23 mg, 23 g, or 23 kg?

b. A pencil: 10 mg, 10 g, or 10 kg?

c. An eyelash: 305 mg, 305 g, 305 kg?

7. Choose the best estimate for the following temperatures.

a. The water temperature for swimming: 22_C, 39_C, or 80_C?

b. A glass of lemonade: 10_C, 5_C, or 40_C?

8. The metric prefixes are also used with measurement of time. If “second” is the fundamental unit of time, what multiple or fraction of a second are the following measurements?

a. Megasecond **b.** Millisecond

c. Microsecond **d.** Kilosecond

e. Centisecond **f.** Picosecond

9. Using the meanings of the metric prefixes, how do the following units compare to a meter? If it exists, give an equivalent name.

a. “Kilomegameter” **b.** “Hectodekameter”

c. “Millimillimicrometer” **d.** “Megananometer”

10. Use the metric converter to complete the following statements.

12. Use the metric converter to answer the following questions.

a. One cubic meter contains how many cubic decimeters?

b. Based on your answer in part (a), how do we move the decimal point for each step to the right on the metric converter?

c. How should we move the decimal point for each step left?

13. Using a metric converter if necessary, convert the following measurements of mass.

a. 95 mg _ _____ cg **b.** 7 kg _ _____ g

c. 940 mg _ _____ g

14. Convert the following measures of capacity.

a. 5 L _ ______ cL **b.** 53 L _ ______ daL

c. 4.6 L _ ______ mL

15. A container holds water at its densest state. Give the missing numbers or missing units in the following table.

a.

b.

c.

16. Convert the following (to the nearest degree).

a. Moderate oven (350_F) to degrees Celsius

b. A spring day (60_F) to degrees Celsius

c. 20_C to degrees Fahrenheit

d. Ice-skating weather (0_F) to degrees Celsius

e._5_C to degrees Fahrenheit

17. By using dimensional analysis, make the following conversions.

a. 3.6 lb to oz

b. 55 mi/hr to ft/min

c. 35 mi/hr to in./sec

d. $575 per day to dollars per minute

18. Prior to conversion to a decimal monetary system, the United Kingdom used the following coins.

1 pound _ 20 shillings 1 penny _ 2 half-pennies

1 shilling _ 12 pence 1 penny _ 4 farthings

(Pence is the plural of penny.)

a. How many pence were there in a pound?

b. How many half-pennies in a pound?

19. One inch is defined to be exactly 2.54 cm. Using this ratio, convert the following measurements.

a. 6-inch snowfall to cm **b.** 100-yard football field to m

20. In performing a dimensional analysis problem, a student does the following:

a. What has the student done wrong?

b. How would you explain to the student a way of checking that units are correct?

a. 1 dm _ _____ mm **b.** 7.5 cm _ _____ mm
c. 31 m _ _____ cm **d.** 3.06 m _ _____ mm
e. 0.76 hm _ _____ m **f.** 0.93 cm _ _____ m
g. 230 mm _ _____ dm **h.** 3.5 m _ _____ hm
i. 125 dm _ _____ hm **j.** 764 m _ _____ km

11. Use the metric converter to complete the following statements.

a. 1 cm^2 _ _____ mm^2 **b.** 610 dam^2 _ _____ hm^2
c. 564 m^2 _ _____ km^2 **d.** 821 dm^2 _ _____ m^2
e. 0.382 km^2 _ _____ m^2 **f.** 9.5 dm^2 _ _____ cm^2
g. 6 540 000 m^2 _ _____ km^2 **h.** 9610 mm^2 _ _____ m^2

GLOSSARY

Rectangle- üçbucaq
increases (decreases)-azalır(artır)
Square- sahə, kvadrat
square desimeter-kvadrat santimetr
metric system-metrik sistem
single square- bir sahə
geometric figures- həndəsi fiqurlar
Consequently- Nəticə etibarilə
is composed- tərtib olunur
condition-şərt, vəziyyət
However- Lakin
subtract the numbers-ədədlərin çıxılması
carried out-həyata keçirilən

GUESTIONS

1. When the unit of measure has been established?
2. What the makers have found the length of the earth's meridian?
3. Where the Microns and millimeters are used only?
4. How many km are 1 geographical miles?
5. How many km are 1 miles of sea?
6. Write Area unit schedule.
7. Write the table of length units.
8. Units of length are also used(complete)

CONCEPT ABOUT NUMBERS

The concept of a number

Comparison of named numbers

Minimizing and Converting Numbers

The ratio of the numbers given by the same unit of measurement and brought to the same unit

Definition 1. A positive number that represents the value of any quantity is called the number of units of a given unit of measure.

Definition 2. If full or decimal places are indicated by the names of the units that produce it, this number is called the number, or the name is not.

If there is a positive integer x denoted by the unit of measure e, then by the first definition, the same number is x*e. Usually when typing numbers, no multiplication is made between the corresponding names and units(can be written in solid).

Let's note that the students measure the length of the classroom and find that it is 6m long. It shows that the 6-meter meter is 6 times the length of the classroom. We write this in mathematical language as :

$$\mathbf{m+m+m+m+m+m = m\cdot 6 = 6m .}$$

If the measurement is taken as the 9/10m length of the table, this can be written as (1/10 piece of m) m*9/10=(9*m)/10 .

Suppose that the unit of measure e is n times in the measured quantity.

Do it write as:

e+e+...+e=e*n=n*e.

If the portion of the e-unit is located in the measured quantity m times, it can be written in mathematical language (the e part):

$$(\frac{1}{n} \text{ pice of e})\cdot\text{m=e}\cdot\frac{m}{n}=\frac{m}{n}e$$

According to the above inscriptions, the title can be described as follows.

Definition 3. Composition of the unit of natural or fractional units, to the measure of the unit of natural or fractional units, is called a named number.

Suppose that parts a and b are equal: a=b. Then $m_e(a)$ the length of a part of line a with the unit of measure e, the $m_e(b)$ length of a part of line b with the unit of measure e, that is:

$$a = b \Leftrightarrow m_e(a) = m_e(b)$$

can.

If a>b, then it is a = b + c for the definition of the magnitude relationship between the fabrics. Lengths of parts- a, b, c and b + c can be lengths as $m_e(a)$, $m_e(b)$, $m_e(c)$ və $m_e(b+c)$ with the unit of measure e . Then

$$a = b + c \Leftrightarrow m_e(a) = m_e(b+c) = m_e(b) + m_e(c)$$

can.

Because of this

$$m_e(a) = m_e(b) + m_e(c)$$

İs correct

$$m_e(a) > m_e(b)$$

On the contrary, it is possible:

$$m_e(b) < m_e(a)$$

Definition 4. If the numbers given by the same unit of units are equal to the actual names of the corresponding names, these numbers are called equal numbers.

Definition 5. If two numbers given by the same unit of measure are larger than the first ones, the first numbers are larger than the second and vice versa.

There are two types of numbers:

1.simple numbers.

2.compound numbers.

Definition 6. Simple units of the same unit of measure are called simple numbers, and numbers of different denominations (units of different measures) are called compound numbers.

For example, 67sm is a simple number and 6dm 7sm is a compound number.

Definition 7. Substituting a numerical bigger unit with a small unit is called as the grinding of the named numbers.

Grinding of the named numbers is made according by the following procedure.

Rules. In order to minimize the given number, it is necessary to multiply the ratio of the corresponding unnamed numerical unit to the smaller unit and write the small unit next to the extract.

For example, to slip 2km, hit 2km for 1000m and write a meter next to it:

2 km =2·(km : m) m = (2·1000) m = 2000 m

Exercise 1. Convert 2 ha 85ar to square meters.

Solution.

2 ha 85 ar = 2·(ha : m^2) m^2 +85·(ar: m^2) m^2 =

=(2·10 000) m^2 +(85·100) m^2 =20 000 m^2 +8500 m^2 =28 500 m^2

Exercise 2. Convert 7 liters of 5 dkliters to per liter.

Solution.

7 kl 5 dkl =7·(kl:l)l+5·(dkl:l)l=(7·1000)l+(5·10)l=7050 l

Exercise 3. Convert 18 days to hours:

Solution.

18 days =18·(day:hour)hour=(18·24)hour=372 hours

Definition. Substitution of the number given by a small unit to the number of the bigger unit called the convertion of the named numbers.

Rule. In order to convert a given number, it is necessary to multiply the ratio of the corresponding unnamed numerical unit to the bigger unit and record the larger unit next to the composition.

Exercise 4. Convert 46 700 m^2 to complex numbers with no more than two names.

Solution.

46 700 m^2 = 40 000 m^2 + 6700 m^2 =

= 40 000 · (m^2 : ha) ha + 6700 · (m^2 : ar) ar =

= (40 000 · 0,0001) ha + (6700·0,01) ar =

= 4 ha + 67 ar = 4 ha 67 ar

Exercise 5. Convert 3,500m to a compound called no more than two names.

Solution.

3500 m = 3000 m + 500 m = 3000·(m:km)km+500 m =

= (3000·0,001) km + 500 m = 3 km 500 m

Exercise 6. Convert 730 liters and 5,500 in complex numbers, with no more than two names.

Solution.

1) 730 l=700 l+ 30 l =700·(l:hl)hl+30 l =

=(700·0,01) hl + 30 l =7 hl +30 l=7hl 30l

2)5300 dm^3=5000 dm^3 +300 dm^3 =5000·($dm^3 : m^3$) m^3 +300 dm^3 =

= (5000·0,001) m^3 +300 dm^3 =5 m^3 300 dm^3

Exercise 7. Convert 240 minutes to an hour.

Solution.

240 min = 240·(min:hour)hour= $\left(240\cdot\frac{1}{60}\right)$hour= 4 hour

ADDITION AND SUBTRACTION OF NAMED NUMBERS.

Definition. A number that denotes the sum of a given number is called the sum of the numbers that characterize the

The process of finding the sum of the given numbers is called the collection of numbers. There may be three cases in the collection of named numbers:

I Collection of simple numbers given by the same unit of measurement.

Rules. In order to collect the simpler numbers given by the same unit of measurement, it is necessary to collect the corresponding unnamed numbers and to write the unit of measure next to the sum obtained.

Example. Find a total of 724 km + 277 km. quantities given.

$$724km+277km=\underbrace{km+km+....+km}_{724}+\underbrace{km+km+....+km}_{277}=$$

$$=\underbrace{km+km+....+km}_{(724+277)}=(724+277)\cdot km=1001km$$

II. Summary of simple numbers given by different measurement units.

Rules. In order to collect simple number units given by different measurement units, either by grinding or rotating, it is necessary to convert them into simple numbers given by the same unit of measure and to collect simple numbers given by the same unit of measure.

Example. Find a total of 3 m + 15 cm.

Solution:

3 m + 15 sm = 300 sm + 15 sm = 315 sm

or

3 m + 15 sm = 3 m + 0,15 m = 3, 15 m

can.

III. Collecting complex numbers.

The collection of complex numbers is brought to the sum of simple numbers given by the same unit of units that produce those numbers.

Example. Find the sum of 5 s 25 kg + 3 s 32 kg.

5 s 25 kq + 3 s 32 kq = (5 s + 25 kq) +(3 s + 32 kq) =

= (5 s +3 s)+(25 kq +32 kq) = 8 s +57 kq = 8 s 57 kq

III.Addition of named numbers.

Definition. The sum of two numbers, and one of these numbers, is called the derivation of numbers for the numerical finding of the other.

There may be three cases when removing numbers called:

I Release of simple numbers given by the same unit of measurement.

Rule. In order to subtract the simpler numbers given by the same unit of measure, it is necessary to subtract the null number corresponding to the declining null number and to write the unit of measurement as the difference.

Task. Calculate 724 km –277 km.

Solution:

$$724\,km-277km=\underbrace{km+km+....+km}_{724}-\underbrace{km+km+....+km}_{277}=$$

$$=\underbrace{km+km+....+km}_{(724-277)}=(724-277)\cdot km=447km$$

II. Release of simple numbers given by different measurement units.

Rules. Simple numbers given by different measurement units must either be subtracted or converted to simple numbers given by the same unit of measurement, and the simpler numbers given by the same unit of units should be deducted.

Example. 3 m -15 sm =300 sm - 15 sm = 285 sm

Solution.

3 m –15 sm = 3 m – 0,15 m = 2,85 m

III. Release of complex numbers.

The derivation of complex numbers is called the simplicity of the numbers given by the same measurement units that produce those numbers.

Example. Find the difference of 5 s 35 kg - 3 s and 32 kg.

Solution :

5 s 35 kq – 3 s 32 kq= (5s–3s)+(35kq–32kq) =2s+3kq=2s 3kq

Or

5 s 35 kq – 3 s 32 kq = 535 kq – 332 kq = 203 kq = 2 s 3 kq

Multiplication and division of named numbers

The multiplication and division of nominal numbers is made by multiplication and division of the unit in terms of natural and fractional numbers.

Definition. Conclusion of the multiplication of the number b to the number ae_1 is equal to the summary of numbers each of which is equal to ae_1.

Example. 7 m ·3 = 7 m +7 m +7 m = 21 m

Rule. For simple numerical numbers, it is necessary to shoot in the appropriate natural number and then write the unit of measurement next to the extract.

Example. 15 kq · 4 = (15· 4) kq = 60 kq

Task. The observer heard a thunderstorm about 14 seconds after the lightning strike. It is known that the sound speed is 330 m / s. Find out how far the lightning strikes the observer.

Solution.

14·330 m = (14·330) m = 4620 m = 4 km 620 m

Definition. Dividing a number into a non-numeric number means that it is obtained by dividing it by dividing the number by the same number.

Rules. In order to divide a simple numeric number, it is necessary to divide the number by the same name and then write the unit of measurement next to the fate obtained.

Definition. Multiplication by a numeric integer means finding a part of that number.

Rules. To draw a simple numeric integer, it is necessary to add the same number to the same number, and to put the unit of measure next to the extract.

Example.

$$5m\cdot\frac{2}{3}=(5m:3)\cdot 2=\frac{5}{3}m\cdot 2=\left(\frac{5}{3}\cdot 2\right)m=3\frac{1}{3}m$$

Rules. In order to shoot in a compound number without the name of a compound, it is necessary to shoot these complex numbers with the same name and to collect the resulting derivatives.

Task. 6 m 32 sm ·2= (6m+32sm)·2=6m ·2+32sm ·2= (6·2)m+(32·2) sm =12 m + 64 sm =12 m 64 sm

Rules. It is enough to divide the corresponding unnamed numbers to find the ratio of the two homogeneous quantities given by the same unit of measurement.

To find the ratio of two homogeneous quantities given by different units of measure, you must first express those quantities by the same unit of measure and find the ratio in the above rule.

Example. 5 m : 2 km = 0,005 km : 2 km =0,0025

İn a different way

5m : 2 km = 5 m : 2000 m =0,0025

QUESTIONS

1.Definition of the named number and example

2. Definition of the none-named number and example

3. Definition of the same named numbers and example

4. Definition of the big or small named numbers and example

5. definition of the simple named numbers and example

6. Definition of the complex(compound) named numbers and example

7. Griding of the named numbers and example

8. Definition about the converting of the named numbers and example

9. Explain the rule about converting of the named numbers

10. Definition about the Addition of the named numbers and example

11. Definition about the subtraction of the named numbers and example

12. Definition of Multiplication of named and none-named numbers

13.Definition of division of named and none-named numbers

14. Rule about division of the simple named number to the none-named number

VARIABLE AND CONSTANT QUANTITIES. DEFINITION OF FUNCTION

Types of quantities

Definitions about functions' elements

Definition of the elementary functions

Examples for elementary functions

In each process occurring in nature and technique, we encounter two types of quantities:

1) constant quantities;

2) changing quantities.

Definition 1. If x is a specified number of clusters of numbers A, it is called a constant number (or simply fixed).

Definition 2. If x is an arbitrary number entered into a cluster of numbers A, it is called a variable number (or simply a variable).

Assume that x and y are arbitrary pairs of values, and each of them is a set of numerical sets.

Definition 3. If each value of x corresponds to a certain value of y, then it is called a functional dependence of the x-value on the value of x.

Definition 4. For each x derived from the X cluster of numbers, the rule that opposes the only number of y from the Y- cluster of numbers is called- the numerical function given in the X cluster of numbers.

x is a variable or function argument, and y is a function of the x argument or y is dependent variable of the x argument. Usually, numeric function is indicated as $y = f(x)$, $y = g(x)$ and etc.

In other wice, of two interrelated variables, if one can give arbitrary values, it is called an arbitrary variable or an argument. If the numerical prices of one of the related quantities change depending on the numerical values of the other, this is called the function of the dependent variable or the other variable.

Definition 5. Set of function values consist of values of argument x which can take in the function $y = f(x)$ are called **the area of the function definition** and all of the numbers of f (x) that the function y receives according to the values of the variable x are called the range of values of the function.

The designated are of the definition of the function f (x) is denoted by D (f) ("define" - in English), and range of values of the function. are denoted by E (f) ("exist" – in english).

Let's assume that the function is given as $y = 2x + 3$, where x takes the variable of [2; 4]. These values of the variable x are called the area of the function definition of the y function: D (f) [2; 4].According to the value of the variable x of [2; 4] , y, the y function take the value of [7; 11]. These values are referred to as the multiplication of the y function: E (f) =[7; 11].

So if the function $y = 2x + 3$ is determined in [2;4] then the range of values of the function will be set of the real numbers [7;11].

The value f (x) in the price is denoted by the symbol f (a) and is called the special value of the function f (x).

For example, when $x = 2$ there is a special function value of the $f(x) = x^2 + 5x + 6$ is $f(2) = 2^2 + 5 \cdot 2 + 6 = 20$.

The power function $(y = x^n),$ exponential function $(y = a^x, a > 0),$ the logarithmic function $(y = \log_a x, a > 0),$ trigonometric functions $(y = \sin x, y = \cos x, y = tg\, x, y = ctg\, x, y = \sec x, y = \cos ec\, x)$ and $y = arc\sin x, y = arc\cos x, y = arctg\, x, arcctg\, x, y = arc\sec x,$ are basic elementary functions.

Definition 6. Functions that are corrected by a finite number of basic operations on basic elementary functions and expressed in a formula are called elementary functions.

For example, $y = \dfrac{1+x}{1-x}, y = 5x^2 \sin x$ etc. The functions are elementary functions. The function $y = \begin{cases} x^3 & x \le 0 \\ x+2 & x > 2 \end{cases}$ is not an elementary function.

QUESTIONS

1. Definition of a fixed number (or simply fixed).
2. Definition of a variable number (or simply a variable).
3. Definition the numerical function given in the X cluster of numbers.
4. Definition of the area of the function definition D(f)
5. Definition of the range of values of the function E(f)
6. Examples for the basic elementary functions.
7. Definition of the elementary functions

GLOSSARY

Function-funksiya

elementary functions- elementar funksiyalar
basic elementary functions.-əsas elementar funksiyalar
exponential function- üstlü funksiya
power function -qüvvət funksiyası
logarithmic function- log funksiyası
trigonometric functions- triqonometrik funksiya
numerical function-ədədi funksiya
changing quantities- dəyişən kəmiyyətlər
constant quantities- sabit kəmiyyyətlər
the area of the function definition- teyin oblasti
range of values of the function- funksiyanın qiymətlər çoxluğu

PRACTICE

1. $y = 3x + 2$ Presented with function formula $x = \{-3; -2; -1; 0; 1\}$. The destination area is numerous. Find the price of the function. Build your schedule.

2. $y = \frac{x^2 - 1}{5}$ Presented with function formula. Find and write the corresponding y-value for each value of x if it is here. $x \in \{-6;\ -5;\ -4;\ 0;\ 5;\ 7\}$

§17. Proportion and properties

Definition. It is said to be in equality of two proportions. Examples of equations we call equality:

1) 2:1=10:5; 2) 2000:20=500:5;

3) 0,5:2=0,75:3; 4) $\frac{3}{4} : \frac{1}{2} = \frac{3}{8} : \frac{1}{4}$.

The ratio should be read as follows: 2 to 1 is equal to 10, 5 is equal to 5 (I). You can read in another way, for example, that 2 is more than 1, and 10 is more than 5.

The numbers included in the equation are called limit limits. So, there are four extremes in the equation. The previous and next limits, that is, the margins on the edges, are called the extremes, and the middle ones are called middle limits.

If a, b, c, d are numbers other than zero and a : b is equal to c :d, then these ratios are proportional. This is the equation

$$a : b=c : d$$

or

$$\frac{a}{b}=\frac{c}{d}$$

write in the form. The a:b=c:d ratio is read as follows: the ratio of a to b is the same as c to d.

Here, a and d are the extremes of the equation, and b and c are the mean limits.

Many different issues are solved with the help of proportions. If there is an equilibrium in the context of the problem, the numbers that are in the equation are the numbers. For example, if an object travels 200m in 10 minutes and 800m in 40 minutes, you can adjust these numbers by:

10 min: 40 min 200 m: 800 m.

There are the following properties of equilibrium.

Property 1. (Main property of equilibrium) The production of the outer limit of the equation is equal to the production of the mean limits.

This property is supported by mathematical symbols

$$\mathbf{a:b =c:d}$$

can be written in the form: a*d =b*c.

Property 2. It is both necessary and desirable that the output of either of these four is equal to the production of the other two, so that the given four numbers can be formed.

This property is supported by mathematical symbols can be written in the form: $(\forall a,b,c,d \in N)[a \cdot d = b \cdot c \Leftrightarrow a:b = c:d]$

Definition. The proportions that are equal to the extremes or the mean limits are called interruptions.

Let's say that the equation is given

$$a:b=b:c.$$

The principal property of the equation is

$$b^2 = a \cdot c.$$

b is called the geometric middle of a and c :

$$b = \sqrt{a \cdot c}.$$

Property 3. The proportions of the proportions of the proportions do not break when the previous (subsequent) ratios are struck or divided by the same number as the zero.

This property can be written with the help of mathematical symbols as follows:

$(\forall n \in N)[a:b=c:d \Rightarrow (a \cdot n):b= (c \cdot n):d \wedge (a:n):b= (c:n):d \wedge a:(b \cdot n) =$
$=c:(d \cdot n) \wedge a:(b:n) =c:(d:n)]$

Property 4. When all the limits of the equation are struck or divided by the same number as zero, the equation does not break.

This property can be written with the help of mathematical symbols as follows:

a) $(\forall n \in N)[a:b=c:d \Rightarrow (a \cdot n):(b \cdot n) = (c \cdot n):(d \cdot n)]$

b) $(\forall n \in N)[a:b=c:d \Rightarrow (a:n):(b:n) = (c:n):(d:n)]$

PROPORTIONS

The concept of proportion

Definition of Proportion

Examples for Proportion

The concept of proportion is useful in solving problems involving ratios.

DEFİNİTİON. Proportion: A proportion is a statement that two given ratios are equal.

The equation 10/12=5/6 is a proportion since 10/12=(5/6)*(2/2)=5/6. Also, the equation 14/21=22/33 is an example of a proportion, since 14 * 33 = 21* 22. In general, a/b=c/d is a proportion if and only if ad = bc. The next example shows how proportions are used to solve everyday problems.

EXAMPLE 1

- Adams School orders 3 cartons of chocolate milk for every 7 students. If there are 581 students in the school, how many cartons of chocolate milk should be ordered?

Set up a proportion using the ratio of cartons to students. Let n be the unknown number of cartons. Then

3(cartons)/7 (students)=n(cartons)/581(students)

Using the cross-multiplication property of ratios, we have that

3*581=7*n co n=(3*581)/7=249

The school should order 249 cartons of chocolate milk.

Commonly used rates include miles per gallon, cents per ounce, and so on. When solving proportions like the one in Example , it is important to set up the ratios in a consistent way according to the units associated with the numbers. In our solution, the ratios 3: 7 and n : 581 represented ratios of cartons of chocolate milk to students in the school. The following proportion could also have been used.

3 (cartons of chocolate milk in the ratio) / n (cartons of chocolate milk in school) = 7(students in the ratio)/ 581(students in school)

Here the numerators show the original ratio. (Notice that the proportion $3/n=581/7$ would not correctly represent the problem, since the units in the numerators and denominators would not correspond.)

In general, the following proportions are equivalent (i.e., have the same solutions). This can be justified by cross-multiplication.

$$a/b=c/d$$
$$a/c=b/d$$
$$b/a=d/c$$
$$c/a=d/b$$

Thus there are several possible correct proportions that can be established when equating ratios.

EXAMPLE 2

A recipe calls for 1 cup of mix, 1 cup of milk, the whites from 4 eggs, and 3 teaspoons of oil. If this recipe serves 6 people, how many eggs are needed to make enough for 15 people?

SOLUTION

When solving proportions, it is useful to list the various pieces of information as follows:

ORIGINAL	RECIPE	NEW RECIPE
Number of eggs	4	x
Number of people	6	15
Thus $4/6=x/15$		

This proportion can be solved in two ways.

CROSS-MULTIPLICATION EQUIVALENT RATIOS

$4/6=x/15$ $4/6=x/15$

$4*15=6*x$

$4/6=(2*2)/(2*3)=2/3=(2*5)/(3*5)=10/15=x/15$

$60=6*x$ Thus $x=10$

$10=x$

Notice that the table in Example 2 showing the number of eggs and people can be used to set up three other equivalent proportions:

$4/x=6/15$ $x/4=15/6$ $6/4=15/x$

EXAMPLE 3

If your car averages 29 miles per gallon, how many gallons should you expect to buy for a 609-mile trip?

SOLUTION

	AVERAGE	TRİP
Miles	29	609
Gallons	1	x

Therefore

$29/1=609/x$ or $x/1=609/29$ Thus x 21.

EXAMPLE 4.

In a scale drawing, 0.5 centimeter represents 35 miles.

a. How many miles will 4 centimeters represent?

b. How many centimeters will represent 420 miles?

SOLUTION

a)	SCALE	ACTUAL
Centimeters	0.5	4
Miles	35	x

Thus,

$$0{,}5/35=4/x .$$

Solving, we obtain x

$$x \ =(35*4)/0{,}5$$

or

$$x=280.$$

B)

	SCALE	ACTUAL
Centimeters	0.5	y
Miles	35	420

Thus,

$$0,5/35=y/420$$

or

$$(0,5*420)/35=y$$

Therefore,

$$y=210/35=6 \text{ centimeters.}$$

GLOSSARY

Therefore-buna görə, odur ki(that’s why)
Thus-beləliklə
Solving-work out-
solve-həll etmək
Obtain-acquire-get-take-əldə etmək
scale –miqyas
drawing-rəsm
Average-ədədi orta(middle,mean,
central, seondary)
Gallon-külliyyat
Expect-be wating, look forward, Wait, Wait for)-gözləmək
Trip-səfər,gəzinti
Notice-elan
xəbərdarlıq,diqqət
(attention,note,
announcement)
involving ratios-nəzərdə tutulan nisbət
Statement-çıxış,bəyanat,təsdiq

Equation-tənlik

QUESTIONS

1. The concept of proportion
2. Definition of Proportion
3. Solve it: Adams School orders 3 cartons of chocolate milk for every 7 students. If there are 581 students in the school, how many cartons of chocolate milk should be ordered?

PROPORTION

A proportion is simply a statement that two ratios are equal. It can be written in two ways: as two equal fractions a/b = c/d; or using a colon, a:b = c:d. The following proportion is read as "twenty is to twenty-five as four is to five."

$$\frac{20}{25}=\frac{4}{5}$$

In problems involving proportions, we can use cross products to test whether two ratios are equal and form a proportion. To find the cross products of a proportion, we multiply the outer terms, called the extremes, and the middle terms, called the means.

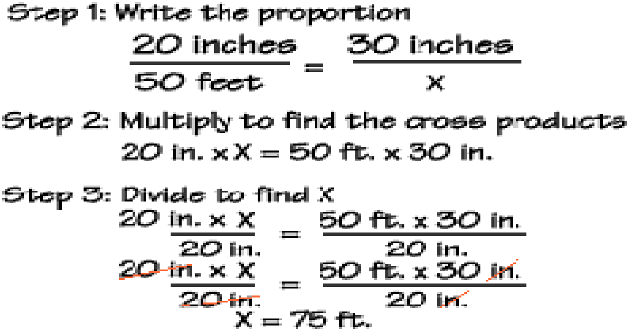
Here, 20 and 5 are the extremes, and 25 and 4 are the means. Since the cross products are both equal to one hundred, we know that these ratios are equal and that this is a true proportion.

$$\frac{20}{25}=\frac{4}{5} \quad \text{cross products: } 20\times 5 = 25\times 4$$

$$100 = 100$$

We can also use cross products to find a missing term in a proportion. Here's an example. In a horror movie featuring a giant beetle, the beetle appeared to be 50 feet long. However, a model was used for the beetle that was really only 20 inches long. A 30-inch tall model building was also used in the movie. How tall did the building seem in the movie?

First, write the proportion, using a letter to stand for the missing term. We find the cross products by multiplying 20 times x, and 50 times 30. Then divide to find x. Study this step closely, because this is a technique we will use often in algebra. We are trying to get our unknown number, x, on the left side of the equation, all by itself. Since x is multiplied by 20, we can use the "inverse" of multiplying, which is dividing, to get rid of the 20. We can divide both sides of the equation by the same number, without changing the meaning of the equation. When we divide both sides by 20, we find that the building will appear to be 75 feet tall.



Step 1: Write the proportion

$$\frac{20 \text{ inches}}{50 \text{ feet}} = \frac{30 \text{ inches}}{x}$$

Step 2: Multiply to find the cross products

$$20 \text{ in.} \times X = 50 \text{ ft.} \times 30 \text{ in.}$$

Step 3: Divide to find X

$$\frac{20 \text{ in.} \times X}{20 \text{ in.}} = \frac{50 \text{ ft.} \times 30 \text{ in.}}{20 \text{ in.}}$$

$$\frac{\cancel{20 \text{ in.}} \times X}{\cancel{20 \text{ in.}}} = \frac{50 \text{ ft.} \times 30 \cancel{\text{in.}}}{20 \cancel{\text{in.}}}$$

$$X = 75 \text{ ft.}$$

Note that we're using the inverse of multiplying by 20-that is, dividing by 20, to get x alone on one side.

In mathematics, the word "proportions" means 2 ratios put into an equation. Some examples of proportions are:

- $^{50}/_{100} = ^{1}/_{2}$
- $^{75}/_{100} = ^{3}/_{4}$
- $^{x}/_{100} = ^{3}/_{4}$, where $x = 75$.

In algebra, proportions can be used to solve many common problems about changing numbers. As an example, for the increase in a $40 purchase of gasoline (petrol), if the price rose 35 cents, from $3.50 to $3.85, the proportion would be:

- $^{x}/_{3.85} = ^{\$40}/_{3.50}$

The solution is simply:

- x = $40/3.50 x 3.85 = $44.00, or $4 more when $0.35 higher.

Many other common calculations can be solved by using proportions to show the relationships between the numbers.

PROPORTIONALITY CONSTANT

A proportionality constant is a number that is used to convert a measurement in one system to the equivalent measurement in another system. For instance, people who are familiar with the traditional system of units used in the United States, pounds, feet, inches, etc., may need to find out the metric equivalent for thesemeasures in grams and meters. To make these calculations they would need some proportionality constants.

One way to write a formula showing how to use a proportionality constant (let us call it "K") is:

$$X*K = Y$$

For instance, people may know that they have 100 eggs and want to know how many dozen eggs they have. The proportionality constant K is then 1 dozen/ 12 eggs.

100 eggs * 1 dozen / 12 eggs = 8 dozen eggs + 4 eggs.

A proportion is usually written as two equivalent fractions. For example:

$$\frac{12\ inches}{1\ foot} = \frac{36\ inches}{3\ feet}$$

Notice that the equation has a ratio on each side of the equal sign. Each ratio compares the same units, inches and feet, and the ratios are equivalent because the units are consistent, and $\frac{12}{1}$ is equivalent to $\frac{36}{3}$.

Proportions might also compare two ratios with the same units. For example, Juanita has two different-sized containers of lemonade mix. She wants to compare them. She could set up a proportion to compare the number of ounces in each container to the number of servings of lemonade that can be made from each container.

$$\frac{40\,ounces}{84\,ounces}=\frac{10\,servings}{21\,servings}$$

Since the units for each ratio are the same, you can express the proportion without the units:

$$\frac{40}{84}=\frac{10}{21}$$

When using this type of proportion, it is important that the numerators represent the same situation – in the example above, 40 ounces for 10 servings – and the denominators represent the same situation, 84 ounces for 21 servings.

Juanita could also have set up the proportion to compare the ratios of the container sizes to the number of servings of each container.

$$\frac{40\,ounces}{10\,servings}=\frac{84\,ounces}{21\,servings}$$

Sometimes you will need to figure out whether two ratios are, in fact, a true or false proportion. Below is an example that shows the steps of determining whether a proportion is true or false.

Example	
Problem	**Is the proportion true or false?**

	$\frac{100 \text{ miles}}{4 \text{ gallons}} = \frac{50 \text{ miles}}{2 \text{ gallons}}$	
miles (check)		The units are consistent across the numerators.
gallons (check)		The units are consistent across the denominators.
	$\frac{100 \div 4}{4 \div 4} = \frac{25}{1}$	Write each ratio in simplest form.
	$\frac{50 \div 2}{2 \div 2} = \frac{25}{1}$	
	$\frac{25}{1} = \frac{25}{1}$	Since the simplified fractions are equivalent, the proportion is true.
Answer	The proportion is true.	

Sometimes you need to create a proportion before determining whether it is true or not. An example is shown below.

Example

Problem	**One office has 3 printers for 18 computers. Another office has 20 printers for 105 computers. Is the ratio of printers to computers the same in these two offices?**

$\frac{\text{printers}}{\text{computers}} = \frac{\text{printers}}{\text{computers}}$	Identify the relationship.
$\frac{3 \text{ printers}}{18 \text{ computers}} = \frac{20 \text{ printers}}{105 \text{ computers}}$	Write ratios that describe each situation, and set them equal to each other.
Printers (check)	Check that the units in the numerators match.
Computers (check)	Check that the units in the denominators match.
$\frac{3 \div 3}{18 \div 3} = \frac{1}{6}$ $\frac{20 \div 5}{105 \div 5} = \frac{4}{21}$	Simplify each fraction and determine if they are equivalent.
$\frac{1}{6} \neq \frac{4}{21}$	Since the simplified fractions are not equal (designated by the ≠ sign), the

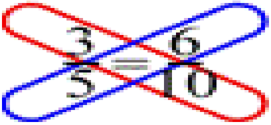
proportion is not true.

Answer The ratio of printers to computers is **not** the same in these two offices.

There is another way to determine whether a proportion is true or false. This method is called "finding the cross product" or "cross multiplying".

To cross multiply, you multiply the numerator of the first ratio in the proportion by the denominator of the other ratio. Then multiply the denominator of the first ratio by the numerator of the second ratio in the proportion. If these products are equal, the proportion is true; if these products are not equal, the proportion is not true.

This strategy for determining whether a proportion is true is called cross-multiplying because the pattern of the multiplication looks like an "x" or a criss-cross. Below is an example of finding a cross product, or cross multiplying.



In this example, you multiply

$$3 \bullet 10 = 30,$$

and then multiply

$$5 \bullet 6 = 30.$$

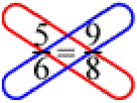
Both products are equal, so the proportion is true.

Below is another example of determining if a proportion is true or false by using cross products.

Example

Problem **Is the proportion true or false?**

$$\frac{5}{6}=\frac{9}{8}$$



Identify the cross product relationship.

$5 \cdot 8 = 40$

$6 \cdot 9 = 54$

Use cross products to determine if the proportion is true or false.

$40 \neq 54$

Since the products are not equal, the proportion is false.

Answer The proportion is false.

Is the proportion $\frac{3}{5}=\frac{24}{40}$ true or false?

A) True

B) False

Show/Hide Answer

Finding an Unknown Quantity in a Proportion

If you know that the relationship between quantities is proportional, you can use proportions to find missing quantities. Below is an example.

Example

Problem	**Solve for the unknown quantity, n.** $\frac{n}{4} = \frac{25}{20}$	
	20 • n = 4 • 25	Cross multiply.
	20n = 100	You are looking for a number that when you multiply it by 20 you get 100.
	$\begin{array}{r} 5 \\ 20\overline{)100} \end{array}$ n = 5	You can find this value by dividing 100 by 20.
Answer	n = 5	

Example		
Problem	**Find the length of a photograph whose width is 10 inches and whose proportions are the same as a 5- inch by 8-inch photograph.**	
	$\frac{width}{length}$	Determine the relationship.

Original photo: $\frac{5 \text{ inches wide}}{8 \text{ inches long}}$	Write a ratio that compares the length to the width of each photograph.
Enlarged photo: $\frac{10 \text{ inches wide}}{n \text{ inches long}}$	Use a letter to represent the quantity that is not known (the width of the enlarged photo).
$\frac{5}{8} = \frac{10}{n}$	Write a proportion that states that the two ratios are equal.
$5 \bullet n = 8 \bullet 10$	Cross multiply.
$5n = 80$	You are looking for a number that when it is multiplied by 5 will give you 80.
$\frac{5n}{5} = \frac{80}{5}$	Divide both sides by 5 to isolate the variable.
$n = \frac{80}{5}$	$\begin{array}{r} 16 \\ 5\overline{)80} \end{array}$
$n = 16$	

Answer	The length of the enlarged photograph is 16 inches.

Now back to the original example. Imagine you want to enlarge a 5-inch by 8-inch photograph to make the length 10 inches and keep the proportion of the width to length the same. You can set up a proportion to determine the width of the enlarged photo.

SOLVING APPLICATION PROBLEMS USING PROPORTIONS

Setting up and solving a proportion is a helpful strategy for solving a variety of proportional reasoning problems. In these problems, it is always important to determine what the unknown value is, and then identify a proportional relationship that you can use to solve for the unknown value. Below are some examples.

Example	
Problem	**Among a species of tropical birds, 30 out of every 50 birds are female. If a certain bird sanctuary has a population of 1,150 of these**

	birds, how many of them would you expect to be female?	
	Let x = the number of female birds in the sanctuary.	Determine the unknown item: the number of female birds in the sanctuary. Assign a letter to this unknown quantity.
	$\frac{30 \textit{ female birds}}{50 \textit{ birds}} =$	Set up a proportion setting the ratios equal.
	$\frac{30 \div 10}{50 \div 10} = \frac{3}{5}$	Simplify the ratio on the left to make the upcoming cross multiplication easier.
	$\frac{3}{5} = \frac{x}{1,150}$	
	3 • 1,150 = 5 • x 3,450 = 5x	Cross multiply.
	$\begin{array}{r} 690 \\ 5\overline{)3,450} \end{array}$ x = 690 birds	What number when multiplied by 5 gives a product of 3,450? You can find this value by dividing 3,450 by 5.
Answer	You would expect 690 birds in the sanctuary to be female.	

Example

Problem	**It takes Sandra 1 hour to word process 4 pages. At this rate, how long will she take to complete 27 pages?**	
	$\frac{4\ pages}{1\ hour} = \frac{27\ pages}{x\ hours}$	Set up a proportion comparing the pages she types and the time it takes to type them.
	$4 \bullet x = 1 \bullet 27$	Cross multiply.
	$4x = 27$	You are looking for a number that when it is multiplied by 4 will give you 27.
	$\begin{array}{r} 6.75 \\ 4\overline{)27.00} \end{array}$	You can find this value by dividing 27 by 4.
	$x = 6.75$ hours	
Answer	It will take Sandra 6.75 hours to complete 27 pages.	

A map uses a scale where 2 inches represents 5 miles. If the distance between two cities is shown on a map as 20 inches, how many miles apart are the two cities?

A) 50 inches

B) 50 miles

C) 8 miles

D) 100 miles

EXAMPLES

find the unknown member of the proportion

1. 35:7=15:x
2. 1(5/2):2(1/3)=x:5(5/6)
3.x:36=4:9
4.2,4:x=1,6:0,5
5. 0,75:3/8=0,3:x
6.3(1/9):6(1/8)=1(11/21):x
7. 7,2:x=1,8:2,5
8.4,8:0,8=x:1(1/6)
9. x:4(3/8)=1,5:1(5/16)
10. 0,38:x=4(3/4):1(7/8)

GLOSSARY

unknown member-məchul ədəd
fabric-parça
Piece-hissə
Scale-çəkmək
Slice-dilimləmək
Pipeline-su borusu
water tap-su kranı

Copper-mis
circular iron-dairəvi dəmir
sand hill-qum təpəsi
Performance-məhsuldarlıq
Mixture-qarışıq
Carpet-xalça
Triangle-üçbucaq
Rectangle-düzbucaq

QUESTONS

1. The concept of ratio
2. Definition of the ratio
3. Complete.... «Ratios allow us to compare ....»
4. Explain: «Ratios can also represent the comparison of part-to-whole or whole - to- part.»
5. Make clear: «Do the ratios represent the same relative amounts?»
6. Carry on this sentence: «Using an argument similar to the one used with fractions, we can show that the ratios.......»
7. DEFİNİTİONEquality of Ratios
8. Complete..... «if n is a nonzero number, then......»
9. Fill in gaps: In the equation a and d are called the....... of the equation a:b = c:d, while b and c are called the ....»

§18. Ratio

The concept of ratio
Definition of the ratio
Examples for a ratio

The concept of ratio occurs in many places in mathematics and in everyday life, as the next example illustrates.

Example (1):

a. In Washington School, the ratio of students to teachers is 17:1, read “17 to 1.”

b. In Smithville, the ratio of girls to boys is 3:2.

c. A paint mixture calls for a 5:3 ratio of blue paint to red paint.

d. The ratio of centimeters to inches is 2.54:1.

In English, the word per means “for every” and indicates a ratio. For example (2), rates such as miles per gallon (gasoline mileage), kilometers per hour (speed), dollars per hour (wages), cents per ounce (unit price), people per square mile (population density), and percent are all ratios.

DEFİNİTİON:Ratio:A ratio is an ordered pair of numbers, written a:b, with $b \neq 0$.

Unlike fractions, there are instances of ratios in which b could be zero. For example (3), the ratio of men to women on a starting major league baseball team could be reported as 9:0. However, since such applications are rare, the definition of the ratio a:b excludes cases in which $b = 0$.

Ratios allow us to compare the relative sizes of two quantities. This comparison can be represented by the ratio symbol a:b or as the quotient a/b.

Quotients occur quite naturally when we interpret ratios. In Example 1 (a), there are 1/17 as many teachers as students in Washington School. In part (b) there are 3/2 as many girls as boys in Smithville. We could also say that there are 3/2 as many boys as girls, or that the ratio of boys to girls is 2:3.

Notice that there are several ratios that we can form when comparing the population of boys and girls in Smithville, namely 2:3 (boys to girls), 3:2 (girls to boys), 2:5 (boys to children), 5:3 (children to girls), and so on. Some ratios give a part-to-part comparison, as in Example 1 (c). In mixing the paint, we would use 5 units of blue paint and 3 units of red paint. (A unit could be any size—milliliter, teaspoon, cup, and so on.) Ratios can also represent the comparison of part-to-whole or whole - to- part. In Example 1. (b) the ratio of boys (part) to children (whole) is 2:5. Notice that the part-to-whole ratio, 2:5, is the same concept as the fraction of the children that are boys, namely 2/5. The comparison of all the children to the boys can be expressed in a whole-to-part ratio as 5:2, or as the fraction 5/2 .

In Example (b), the ratio of girls to boys indicates only the relative sizes of the populations of girls and boys in Smithville. There could be 30 girls and 20 boys, 300 girls and 200 boys, or some other pair of numbers whose ratio is equivalent.

It is important to note that ratios always represent relative, rather than absolute, amounts. In many applications, it is useful to know which ratios represent the same relative amounts. Consider the following example.

Example:

In class 1 the ratio of girls to boys is 8:6. In class 2 the ratio is 4:3. Suppose that each class has 28 students. Do these ratios represent the same relative amounts?

Notice that the classes can be grouped in different ways:

- **Class 1: GGGG GGGG**
- **BBB BBB**
- **Class 2: GGGG**
- **BBB Ratio 8:6**

- **Ratio4:3**

Figure 1.

The subdivisions shown in Figure 1 do not change the relative number of girls to boys in the groups. We see that in both classes there are 4 girls for every 3 boys.

Hence we say that, as ordered pairs, the ratios 4:3 and 8:6 are equivalent, since they represent the same relative amount. They are equivalent to the ratio 16:12.

From last Example it should be clear that the ratios a:b and ar:br, where $r \neq 0$, represent the same relative amounts.

Using an argument similar to the one used with fractions, we can show that the ratios a:b and c:d represent the same relative amounts if and only if ad = bc. Thus we have the following definition.

DEFİNİTİON Equality of Ratios: Let a/b and c/d any two ratios. Then a/b=c/d if and only if ad = bc.

Just as with fractions, this definition can be used to show that if n is a nonzero number, then (a*n/b*n)=a/b or an:bn =a:b. In the equation a and d are called the extremes, since a and d are at the "extremes" of the equation a:b = c:d, while b and c are called the means. Thus the equality of ratios states that two ratios are equal if and only if the product of the means equals the product of the extremes.

GLOSSARY

Represent	təqdim etmək
Extreme	kənardakı
mean	orta
Ratio	nisbət
nonzero number	≠0

relative amounts- nisbi miqdar
Subdivisions-bölmələr
Hence-deməli, buradan(from here, from this place)
Suppose-fərz etmək, güman etmək
to note-qeydə almaq
Comparison-müqayisə(noun)
Several- bir neçə
same concept- eyni konsepsiya
can be expressed-ifadə oluna bilər
Population-əhali(inhabitants)
To be an ordered-sifariş olunmaq
İndicate-göstərmək, ifadə etmək(show,display, express)
Per-(every)- hər bir (each)
people per square mile-hər m^2 düşən insanlar
Major-əsas(main, basic)-şəhər meri
Exclude-xaric etmək
İnclude- daxil etmək
İnterpret-(explain,make clear)-izah etmək
a part-to-part-hissə-hissə
a whole-to-part- tamın hissəsi
Product- hasil

EXAMPLES

1. Find the ratio of the following numbers:
   a) 15 and 5;
   b) 0,5 and 0,125
   c) 1,2 and 1.44;
   d) 3,2 and 9,6;
   e) 5 and 3/5;

2. If the ratio is the previous limit 7 (7/18) and the ratio is equal to 2/7, find the next limit.

3. If the relative limit is 4.5 and the ratio is 0.9, find the previous limit.

4. Simplify the following ratios:

a) 1) 45: 135; 2) 25: 75; 3) 150: 450; 4) 36: 144; 5) 45: 36; 6) 66: 165;

b) 7) 408: 188; 8) 96: 102.4; 9) 60: 180;

c) 10) 18: 45; 11) 1125: 1225; 12) 5.4: 91.8

5. Find the ratio.

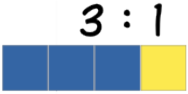
a) 2/15 in to 2;

b) 2/15 in to 5/8;

c) 2/15 in to 5/6;

d) 2/15 in to 2 (2/5);

A ratio compares values**.** A ratio says how much of one thing there is compared to another thing.

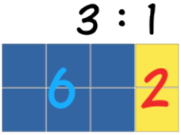


There are 3 blue squares to 1 yellow square

Ratios can be shown in different ways:

Use the ":" to separate the values:	3 : 1
Or we can use the word "to":	3 to 1
Or write it like a fraction:	3**1**

A ratio can be scaled up:



Here the ratio is also 3 blue squares to 1 yellow square, even though there are more squares.

Using Ratios

The trick with ratios is to always multiply or divide the numbers **by the same value**.

Example:

4 : 5 is the same as **4**×2 **: 5**×2 **= 8 : 10**

Recipes

Example: A Recipe for pancakes uses 3 cups of flour and 2 cups of milk.

So the ratio of flour to milk is **3 : 2**

To make pancakes for a LOT of people we might need 4 times the quantity, so we multiply the numbers by 4:

3×4 **: 2**×4 **= 12 : 8**

In other words, 12 cups of flour and 8 cups of milk.

The ratio is still the same, so the pancakes should be just as yummy.

"Part-to-Part" and "Part-to-Whole" Ratios

The examples so far have been "part-to-part" (comparing one part to another part).

But a ratio can also show a part compared to the **whole lot**.

Example: There are 5 pups, 2 are boys, and 3 are girls

Part-to-Part:

The ratio of boys to girls is 2:3 or $^{2}/_{3}$

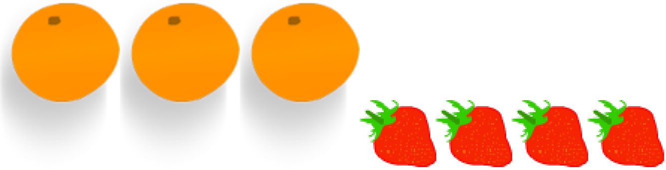
The ratio of girls to boys is 3:2 or $^{3}/_{2}$

Part-to-Whole:

The ratio of boys to **all** pups is 2:5 or $^{2}/_{5}$

The ratio of girls to **all** pups is 3:5 or $^{3}/_{5}$

Try It Yourself



What is the ratio of oranges to strawberries? ☐ : ☐ ✓

What is the ratio of strawberries to oranges? ☐ : ☐ ✓

What is the ratio of oranges to total fruit? ☐ : ☐ ✓

What is the ratio of strawberries to total fruit? ☐ : ☐ ✓

Scaling

We can use ratios to scale drawings up or down (by multiplying or dividing).

The height to width ratio of the Indian Flag is **2:3**

So for every **2** (inches, meters, whatever) of height there should be **3** of width.

If we made the flag 20 inches high, it should be 30 inches wide.

If we made the flag 40 cm high, it should be 60 cm wide (which is still in the ratio 2:3)

Example: To draw a horse at 1/10th normal size, multiply all sizes by 1/10th

This horse in real life is 1500 mm high and 2000 mm long, so the ratio of its **height to length** is

1500 : 2000

What is that ratio when we draw it at 1/10th normal size?

$$1500 : 2000 = 1500\times\mathbf{1/10} : 2000\times\mathbf{1/10}$$

$$= \mathbf{150 : 200}$$

We can make any reduction/enlargement we want that way.

Big Foot?

Allie measured her foot and it was 21cm long, and then she measured her Mother's foot, and it was 24cm long.

"I must have big feet, my foot is nearly as long as my Mom's!"

But then she thought to measure heights, and found she is 133cm tall, and her Mom is 152cm tall.

In a table this is:

	Allie	Mom
Length of Foot:	21cm	24cm
Height:	133cm	152cm

The "foot-to-height" ratio in fraction style is:

Allie: 21**133** Mom: 24**152**

So the ratio for Allie is **21 : 133**

By dividing both values by 7 we get **21/7 : 133/7** = **3 : 19**

It is still the same ratio, right? Because we divided both numbers by the same amount.

And the ratio for Mom is **24 : 152**

By dividing both values by 8 we get **24/8 : 152/8** = **3 : 19**

This time we divided by 8, but that ratio stays the same, too.

The simplified "foot-to-height" ratios are now:

Allie: 3**19** Mom: 3**19**

"Oh!" she said, "the Ratios are the same".

"So my foot is only as big as it should be for my height, and is not really too big."

QUESTIONS

1. The concept of ratio
2. Definition of the ratio

3. Complete.... «Ratios allow us to compare ....»

4. Explain: «Ratios can also represent the comparison of part-to-whole or whole - to- part.»

5. Make clear: «Do the ratios represent the same relative amounts?»

6. Carry on this sentence: «Using an argument similar to the one used with fractions, we can show that the ratios.......»

7. DEFİNİTİON

Equality of Ratios

8. Complete..... «if n is a nonzero number, then......»

9. Fill in gaps: In the equation a and d are called the....... of the equation a:b = c:d, while b and c are called the ....»

§19. Scale

Definition of proportions

Properties of proportions

About Scale

Definition. It is said to be in equality of two proportions.

Examples of equations we call equality:

1) 2:1=10:5;

2) 2000:20=500:5;

3) 0,5:2=0,75:3;

4) $\frac{3}{4}:\frac{1}{2}=\frac{3}{8}:\frac{1}{4}$

The ratio should be read as follows: 2 to 1 is equal to 10, 5 is equal to 5 (I). You can read in another way, for example, that 2 is more than 1, and 10 is more than 5.

The numbers included in the equation are called limits of means and extremes. So, there are 2 extremes in the equation. The previous and next limits, that is, the margins on the edges, are called the extremes, and the middle ones are called means (middle) limits.

Property 1: If a, b, c, d are numbers other than zero and a : b is equal to c :d, then these ratios are proportional. This is the equation

$$a : b=c : d$$

write in the form:

$$\frac{a}{b}=\frac{c}{d}$$

The

$$a:b=c:d$$

ratio is read as follows: the ratio of a to b is the same as c to d. Here, a and d are the extremes of the equation, and b and c are the mean limits. Many different issues are solved with the help of proportions. If there is an proportion in the context of the problem, the numbers that are in the equation are the numbers. For example, if an object travels 200m in 10 minutes and 800m in 40 minutes, you can adjust these numbers by:

10 min: 40 min 200 m: 800 m.

Property 2. It is both necessary and desirable that the output of either of these four is equal to the multiolication of the other two, so that the given four numbers can be formed.

This property is supported by mathematical symbols can be written in the form:

$$(a,b,c,d€N)[a*d=b*c \qquad a:b=c:d]$$

Definition. The proportions that are equal to the extremes or the mean limits are called interruptions. Let's say that the equation is given a:b=b:c.The principal property of the equation $b^2=a*c$ b is called as geometric middle of a and c:

$$b=\sqrt{a\cdot c}$$

Property 3. The proportions of the proportions do not break when the previous (subsequent) ratios are struck or divided by the same number as the zero. This property can be written with the help of mathematical symbols as follows:

$$(\forall n\in N)[a:b=c:d\Rightarrow(a\cdot n):b=(c\cdot n):d\wedge(a:n):b=(c:n):d\wedge a:(b\cdot n)=$$
$$=c:(d\cdot n)\wedge a:(b:n)=c:(d:n)]$$

Property 4. When all the limits of the equation are struck or divided by the same number as zero, the equation does not break.

This property can be written with the help of mathematical symbols as follows:

a) $(\forall n\in N)[a:b=c:d\Rightarrow(a\cdot n):(b\cdot n)=(c\cdot n):(d\cdot n)]$

b) $(\forall n\in N)[a:b=c:d\Rightarrow(a:n):(b:n)=(c:n):(d:n)]$

Now talk about Scale.

Measurement scale, in statistical analysis, the type of information provided by numbers. Each of the four scales (i.e., nominal, ordinal, interval, and ratio) provides a different type of information. Measurement refers to the assignment of numbers in a meaningful way, and understanding measurement scales is important to interpreting the numbers assigned to people, objects, and events.

Nominal Scales

In nominal scales, numbers, such as driver's license numbers and product serial numbers, are used to name or identify people, objects, or events. Gender is an example of a nominal measurement in which a number (e.g., 1) is used to label one gender, such as males, and a different number (e.g., 2) is used for the other gender, females. Numbers do not mean that one gender is better or worse than the other; they simply are used to classify persons. In fact, any other numbers could be used, because they do not represent an amount or a quality. It is impossible to use word names with certain statistical techniques, but numerals can be used in coding systems. For example, fire departments may wish to examine the relationship between gender (where male = 1, female = 2) and performance on physical-ability tests (with numerical scores indicating ability).

Ordinal Scales

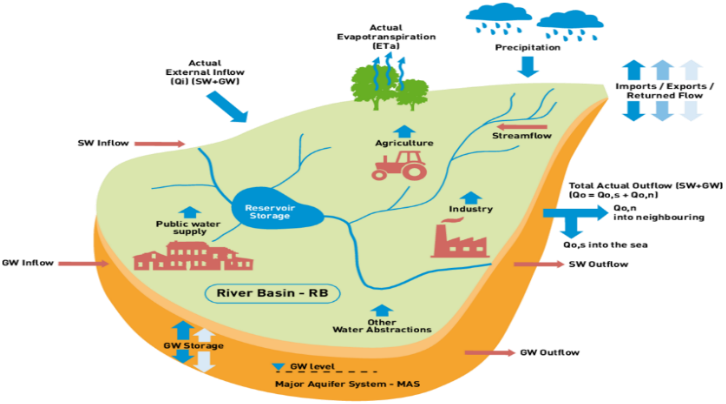
In ordinal scales, numbers represent rank order and indicate the order of quality or quantity, but they do not provide an amount of quantity or degree of quality. Usually, the number 1 means that the person (or object or event) is better than the person labeled 2; person 2 is better than person 3, and so forth—for example, to rank order persons in terms of potential for promotion, with the person assigned the 1 rating having more potential than the person assigned a rating of 2. Such ordinal scaling does not, however, indicate how much more potential the leader has over the person assigned a rating of 2, and there may be very little difference between 1 and 2 here. When ordinal measurement is used (rather than interval measurement), certain statistical techniques are applicable (e.g., Spearman’s rank correlation).

Interval Scale

In interval scales, numbers form a continuum and provide information about the amount of difference, but the scale lacks a true zero. The differences between adjacent numbers are equal or known. If zero is used, it simply serves as a reference point on the scale but does not indicate the complete absence of the characteristic being measured. The Fahrenheit and Celsius temperature scales are examples of interval measurement. In those scales, 0 °F and 0 °C do not indicate an absence of temperature.

Ratio Scales

Ratio scales have all of the characteristics of interval scales as well as a true zero, which refers to complete absence of the characteristic being measured. Physical characteristics of persons and objects can be measured with ratio scales, and, thus, height and weight are examples of ratio measurement. A score of 0 means there is complete absence of height or weight. A person who is 1.2 metres (4 feet) tall is two-thirds as tall as a 1.8-metre- (6-foot-) tall person. Similarly, a person weighing 45.4 kg (100 pounds) is two-thirds as heavy as a person who weighs 68 kg (150 pounds).



GLOSSARY

Geometric middle-həndəsi orta
İnterruption-fasilə,interval,pauza
Previous adj-öncəki, ötən(last, past)
Subsequent adj-sonrakı
(next, following, incoming)
Principal-adj-əsas (main, base, primary, essential) -noun rəis
Desirable-adj-istəməli, arzuolunan

(loveable, lovable)
Necessary-lazımlı –(adj)- (significant, important,useful)
Issue- məsələ, problem,nəşr
(task, example) noun
Equation- tənlik(noun)
Margins-haşiyə, sahənin kənarı,sahələr
Mean- orta(middle,medium)
Extreme- kənar (last, outside)
Output-exit-çıxış
Adjust-tənzimləmək,
uyğunlaşdırmaq, düzəltmək
In another way- başqa yolla
(in different way)
unknown member-məchul ədəd
fabric-parça
Piece-hissə
Scale-çəkmək
Slice-dilimləmək
Pipeline-su borusu
water tap-su kranı
Copper-mis
circular iron-dairəvi dəmir
sand hill-qum təpəsi
Performance-məhsuldarlıq
Mixture-qarışıq Carpet-xalça
Triangle-üçbucaq
Rectangle-düzbucaq

EXAMPLES

1. Find the unknown member of the proportion. They were given 25.2 manat for 6 m of fabric. How much should you pay for such a 3 meter piece of fabric? How much should you pay for the 18m piece of fabric; for kind of 12 m; ?

2. When you scale a live cow, an average of 42 s of meat is sliced every 67 s. How many kilograms of meat can be obtained from a cow weighting 335 kg?

3. 7.83 kg of copper is required to pour 30 pipeline water taps. How much copper is needed to pour such 100 water taps?

4. It is necessary to cut 2 kg iron from a circular iron weight of 8.75 kg and a length of 3.5 m. What is the length of the piece of cut ?

5. The machine made 1550 m³ of sand hill in 5 hours and 10 minutes. How many cubic meters of sand can make up to 8 hours 16 minutes if the performance of this machine does not change?

6. The train has got velocity 42 km / h and takes 8 hours 32 minutes. How many hours does the train travel with the velocity 56 km / h for getting e same distance?

7. The 1.5 m wheel had 96 rounds at some distance. How long does a 2.4 m wheel drive at that distance?

8. There is 50 g salt in 800 g mixture . How much salt is there in the 240 g; 1 kq; 20 q; mixture?

9. If 12 workers are making a carpet for 30 days, how many days does 9 workers make the same carpet?

10. The sides of the triangle with a perimeter 40 m directly proportional to numbers 2, 3 and 5. Find the length of the largest side.

11. The sides of a rectangle directly proportional to numbers 2; 3; 4 and 6. If the smallest side is 12 cm, find the perimeter of the rectangle.

QUESTIONS

1. Definition of proportions

2. Propertie 1 of proportion
3. Propertie 2 of proportion
4. Propertie 3 of proportion
5. Propertie 4 of proportion

PREFERENCE

Multi-Task Setting Involving Simple and Complex Tasks: An Exploratory Study of Employee Motivation Maia Jivkova Farkas University of South Florida, maia@usf.edu 2014

§20. Simple and complex issues.

In order to facilitate the students' improvement of their skills such as reasoning, representing, modelling and communication, word problems are set to work during their formal education. By this way, students can gain experience by using their mathematical knowledge and skills in daily life (Greer, 1997; Reusser & Stebler, 1997). In other words students benefit from mathematical word problems about building connection between conceptual and mathematical knowledge. Reusser and Stebler (1997) stated that these problems prepare suitable environment for students to provide language formation, reasoning and mutual interaction.

According to the literature (Carpenter, Carey & Kouba,1990), Carpenter and Moser, 1981; Fuson, 1992;Van de Walle, Karp, and Williams, 2010), the * Author: gkamuran@cu.edu.tr Çukurova Üniversitesi Eğitim Fakültesi Dergisi Vol: 46 No: 2 Sayfa: 639-648 www.cufej.com Kamuran TARIM – Çukurova Üniversitesi Eğitim Fakültesi Dergisi, 46(2), 2017, 639-648 640 word problems linked with addition and subtraction considering the relationship on which they were built. In word problems, children's performances on problem solving skills change in line with what the unknown is (Haylock & Cockburn, 2004; Sarama & Clements, 2009) .In an addition or in a sub traction, as the position of the unknown change(2+3=?; ?+3=5; 2+?=5; 5-2=?; 5-?=3…) the type of the word problem change accordingly. Van de Walle, Karpand Williams (2016) considered all word problems classification as a whole and evaluated them. Van De Walle et al. (2016) have grouped addition and subtraction problems as change (Join and separate), part-part whole problems, and compare problems. The problems were grouped into sub-categories when one of these elements was unknown. The descriptions of this classification and sub-categories were as follows: Join/Add to problems: Three amounts can be seen in this kind of problems: the initial, the change (the part being added or joined) and the result (the amount that is obtained after the operation). It is generally preferred not to give one of these elements as the unknown in the problem. Separate: In these problems, the change is that an amount is being physically removed or taken away from the start value.

The sub-categories and related examples of Join and Separate problems are in Table 1. Table 1. Change (Join and Separate) problems (Adapted from Van De Walle et al. (2016) p.193) Result unknown Change Unknown Start Unknown Join/Add to Berke had 4 apples. Ege gave him 3 more. How many apples does Berke have altogether? 4+3=? Berke had 3 apples. Ege gave him some more. Now Berke has 7 apples. How many did Ege give him? 3+?=7 Berke had some apples. Ege gave him 3 more. Now Berke has 8 apples. How many apples did Berke have to begin with? ?+3=8 Separate/ take from Berke had 8 apples. He gave 4 apples to Ege. Now How many apples does Berke have now? 8-4=? Berke had 9 apples. He gave some to Ege. Now Berke has 4apples. How many did he give to Ege? 9-?=4 Berke had some apples. He gave 4 to Ege. Now Berke has 3 apples left. How many apples did Berke have to begin with? ? - 4=3 Part-part-whole problems: These problems involve two parts that are conceptually or mentally combined into one collection or whole. The combining may be a physical action or a mental combination in which the parts are not physically combined. Compare problems: Compare problems involve the comparison of two quantities. The third amount is not actually present but is the difference between the two amounts. However, the third amount is the difference between the two already-given amounts.

The subcategories and the related examples of Part-part Whole problems and Compare problems are as follows (Table 2): Kamuran TARIM – Çukurova Üniversitesi Eğitim Fakültesi Dergisi, 46(2), 2017, 639-648 641 Table 2. Part-Part Whole and Compare Problems (Adapted from Van De Walle et al. (2016) p.193) Whole unknown One-part Unknown Part-part Whole Ege has 3 apples and 5 oranges. How many fruits does he have? Ege has 8 fruits. Three of his fruits are apples, and the rest are oranges. How many oranges does Ege have? 3+5=? 8=3+? or 8-3=? Difference Unknown Larger Unknown Smaller Unknown Compare Ege has 7 apples and Berke has 4 apples. How many more apples does Ege have than Berke? 7-3=? Egehas 4 more apples than Berke. Berke has 12 apples. How many apples does Ege have? 8+4=? Ege has 3 more apples than Berke. Ege has 6 apples. How many apples does Berke have? ?+3=6 Researchers mention that children should be taught different types of structures to define the important characteristics of the problems and to determine in what context they should addorsubtract (Fagnant & Vlassis, 2013; Fuchs, Fuchs, Prentice, Hamlett, Finelli, & Courey, 2004; Xin, Jitendra, & Deatline-Buchman, 2005; Cited in Van De Walle et al. 2016). In otherwords; they say that these types should be presented to children, so that they solve word problems successfully (Sarıbaş & Arnas, 2017). The problems presented to children are mostly the ones in the text books and work books. The studies show that textbooks do not include some problem types or do not give equal importance to all (Parmjit & Teoh, 2010; Parmjit, 2006; Despina & Herikleia, 2014; Olkun & Toluk, 2002). When the related literature was reviewed, it is seen that the researches on mathematical word problems were mostly carried out with the children in the pre-school period (Altun, Dönmez, İnan&Özdilek, 2001; Artut, 2015; Carpenter, Hiebert, & Moser, 1983; Carpenter, Franke, Ansell, Fennema, & Weisbeck, 1993; Davis & Pepper, 1992; Manches, O'Malley&Benford,2010; Monroe & Panchyshyn, 2005; Patel & Canobi, 2010; Tarım, 2009;Saribaş&AktaşArnas, 2016). Olkun and Toluk (2002) conducted a study at primary school level. They investigated the elementary school students' achievements on mathematical word problems and the distribution of these problems in the course books. It was revealed by their findings that the students' achievements in terms of problem types displayed similarities with the distribution of the problem types in the course books. Textbooks are so central in teaching that using textbook strongly influences what teachers do. The teacher is much more likely to have math lessons that link important mathematics concepts to contexts that engage students (Van de Walle et al., 2010). Therefore, the contents of the word problems in textbooks, the numbers that are used in these problems and the problems types presented to the students in textbooks have significant effects on their establishment of contextual relationships between the operations. The purpose of this study is twofold: One is to determine student success on the mathematical word problems based on addition and subtraction. The other is to examine whether elementary school mathematics textbooks adequately include the standard word problems representing different meanings of addition and subtraction. Kamuran TARIM – Çukurova Üniversitesi Eğitim Fakültesi Dergisi, 46(2), 2017, 639-648 642 Significance: The studies for making the 1st and 2nd grade elementary school students comprehend the addition and subtraction are presented intensively. The studies about multiplication and division are added in higher (the 3rd and 4th) grade. Therefore, it was found acceptable to work with 3rd grade elementary school students in order to determine at what level and at what types of word problems based on addition and subtraction the success is achieved. In Turkey, textbooks and workbooks at all levels of elementary education are designed by the committees which were assigned by the Ministry of National Education and they are distributed to all elementary schools (both private and state schools) free of charge. All schools are obliged to use these textbooks in their programs. Accordingly, it becomes important to identify the distribution of the problems in these textbooks which are compulsory to follow, to identify how much of these problems which were designated in the literature was used and to present the existing situation on this issue. The results obtained from this study can provide important contributions to the process of re-evaluation of the textbooks and to the studies which aim to determine the reasons of the failure of students on some problem categories. Method In this research, quantitative data was obtained by descriptive survey model in order to find at which mathematical word problems the students are successful and qualitative data was obtained by document analysis method in order to determine the distribution of the types of word problems which are presented to the students in mathematics textbooks and workbooks. For this reason, mixed methods (convergent parallel design) was used in the study. The convergent parallel design gives equal importance to qualitative and quantitative methods and analyses them separately, then combines the results in the phase of the interpretation (Creswell, 2014). Participants The study was conducted with the 3rd grade students attending to 3 elementary schools which represent low, middle and high socio-economic levels in Adana, which is a metropolis in the south of Turkey. 158 students participated in the research. It was specified that 45 of them were from low socioeconomic level, 62 of them were from middle socio-economic level and 51 of them were from high socio-economic level. The textbooks and the workbooks which were being used in elementary school mathematics lessons were used as the second data source. The mathematics textbooks in Turkey are designed in forms of a textbook and a workbook. In this study, these two books were included in the research in all grade levels. In other words, totally 6 books were investigated as the textbooks and the workbooks for the first, second and third grade levels. Instrument In the research, the quantitative data consisted of 13 problems about each category and their subcategories on word problems with addition and subtraction. The problems were prepared and formed by the researcher and they are constituted from word problems about addition and subtraction the classification of which was made by Van De Walle et al. (2010). The reliability of the test was found as 0.77 from KR-20 measurement. As this value was higher than 0.70, the test was accepted as reliable. So as to obtain qualitative data, the textbooks which were distributed by Ministry of National Education (MoNE) and were being used in the 1st, 2nd and 3rd grades of elementary schools were acquired. These books were investigated at all grade levels one by one by considering the categories which were expressed in the literature (Van De Walle et al., 2010). The number of the problems in each category was recorded by writing the number of the pages they exist next to them. After that, the number of the problems in each category was found.

The quantitative data obtained from the students in this study was analyzed by specifying frequency and percentage values. The document analysis method was used in the analysis of mathematics textbooks and workbooks. For this purpose, the word problem categorization which was formed by Van De Walle et al. (2010) was taken as a basis and a "Record Form" was created. The kinds of word problems in the books were examined by the researcher and they were recorded to this form. Then, the frequencies and percentages were calculated. Findings and Comments While presenting the findings obtained from the study, the descriptive statistics of the students' answers given to the word problems about addition and subtraction (Table 3) were given and commented. After that, the data and comments on how the distribution of the word problem types in mathematics textbooks and workbooks is were presented. Descriptive statistics of the students' answers given to the word problems (n=158) PROBLEMS Correct f % Problem1: Join: Result Unknown 8+7=? 150 94.9 Problem2: Join: Change Unknown 17+?=25 107 67.7 Problem3: Join: Initial Unknown ?+16=25 104 65.8 Problem4: Part-Part-Whole: Whole Unknown 26+34=? 148 93.7 Problem5: Part-Part-Whole: Part Unknown 19+?=46 102 64.6 Problem6: Separate: Result Unknown: 72-45=? 112 70.9 Problem7: Separate: Change Unknown: 14-?=9 119 75.3 Problem8: Separate: Initial Unknown ?-22=79 102 65.2 Problem9: Compare: Difference Unknown 44-29=? 94 59.5 Problem10: Compare: Difference Unknown 56-34=? 118 74.7 Problem11: Compare: Smaller Unknown 27-7=? 99 62.7 Problem12: Compare: Larger Unknown 12+9=? 129 81.6 Problem13: Compare: Larger Unknown 18+5=? 67 42.4 When the students' answers given to the questions were analyzed

(Table 3), it was seen that 94.9 % of the students gave correct answers to the problem 1 and 93.7 % of the students answered the problem 4 correctly. The students were able to solve both types of problems. These problems were Join: result unknown (Problem 1) and Part-Part-Whole: Whole Unknown (Problem 4) which require the operation of addition. These types of problems especially Join: Result Unknown problems appear in the textbooks the most (28.77 %). It can be said that Part-Part-Whole: Whole Unknown problems (Problem 4) are found easy by students as they show similarities with Join: Result Unknown problems and they require the operation of addition. When Part-Part-Whole problems are considered, the students showed less success in the problem in which part was unknown (19+?=46) than in the problem in which whole was unknown. Here, the students had more difficulty in solving the question which requires the operation of subtraction than the other question. The reason of this difference may derive from the fact that the unknown is in the middle and the student is searching for an answer for 19+x=46. No matter what type the problems were, the students displayed similar distribution in initial unknown problems (?+16=25; ?-22=79) (Join: Initial Unknown: 65.8 %; separate: Initial unknown: 65.2 %). According to Van de Walle et al. (2010, p.148), the join or separate problems with initial unknown are considered as one of the most difficult ones. They said that the children modelling the problems exactly are not aware of the number of counters to begin with.

Also, problems in which the change amounts are unknown are difficult. The document analysis revealed that these types of problems were encountered merely in the textbooks and workbooks (Join: Initial Unknown: 0.71%; Separate: Initial Unknown: 5.75 %).The relatively low success which was observed here can be considered natural as these types of problems exhibit difficult structure and they are the ones which the students encounter. Van de Walle (2001, p.148) mentioned that in most books, there was simply more emphasis on join and separate problems with result-unknown structure. One of the reasons why the children failed in the other problem categories may stem from their infrequent exposure to these other problems. Compare: When difference unknown problems (Problems 9 and 10) were analyzed, it was seen that the success which was showed in these problems was quite different although they were the same type.

When the reason of this difference was questioned, it drew attention that there were key words in their explanations such as “more” and “fewer”. To explain better these two problems, they were given below: PROBLEM9: Ayla has got 29 candies whereas her sister Elif has got 44 candies. How many more candies has Elif got than Ayla? (Compare: Difference Unknown 44-29=?) PROBLEM10: Cemre has got 56 stick crackers whereas Selim has got 34 stick crackers. How many fewer stick crackers has Selim got than Cemre? (Compare: Difference Unknown 56-34=?) As it can be seen, although the key word “more” which associates with the operation of addition was used in problem 9, the operation of subtraction is required for the solution of the problem as opposite.

Therefore, the students might have had more difficulties in solving this problem than the other one (59.5 % vs. 74.7 %). Similarly, the students showed different achievement level even though the problems 12 and 13 were the same type; Compare: Larger Unknown (for problem 12:81.6 %; for problem 13: 42.4 %). Consequently, it was thought that the key words “more” and “fewer” which were used in the explanation of the problems were effective in this difference. These problems are as follows:

PROBLEM12: Meryem has got 12 pullovers. As Ela has got 9 more pullovers than Ela, how many pullovers has Ela got? (Compare: Larger Unknown: 12+9=?)

PROBLEM13: Nurcan has bought 18 bananas at the greengrocer's. Nurcan has got 5 fewer bananas than Duygu. How many bananas has Duygu got? (Compare: Larger Unknown 18+5=?) As it can be seen, the key word "fewer" misled the students in the solution of Problem 13. In this problem; the word "fewer" was associated with subtraction. However, it was necessary to add 18 and 5 to solve the problem. In other words, the word "fewer' misguided the students. In this study, document analysis was conducted to see to what extent each type of problems was presented to students in textbooks and workbooks. The distribution of word problems in Mathematics textbooks and workbooks according to the categories.

Result unknown problems appeared in the textbooks more (28.77 % + 27.33% = 56.10%). There were scarcely any initial unknown problems. Similarly, it was observed that all sub-categories of compare problems were scarcely used in the first three years of elementary education, especially larger unknown and smaller unknown problems. Similar to this, in a study by Olkun and Toluk in Turkey (2002) found out that all types of word problems were not included in the primary Categories Subcategories 1 st grade 2 nd grade 3 rd grade Total f % f % f % f % Join Problems Result unknown 17 15.47 18 50 5 41.66 40 28.77 Change unknown 13 14.28 2 5,55 - 15 10.79 Initial unknown - - 1 2.77 - 1 0.71 Separate Problems Result unknown 26 28.57 10 27.77 2 16.66 38 27.33 Change unknown 4 4.39 - 2 16.66 6 4.31 Initial unknown 5 5.49 3 8.3 - 8 5.75 Part-part Whole Problems Whole unknown 15 16.48 - 1 8.33 16 11.51 Part unknown 6 6.59 - - 6 4.31 Compare Problems

Difference unknown 5 5.49 2 5.55 2 16.66 9 6.47 Larger unknown - - - - Smaller unknown - - - - Total 91 100 36 100 12 100 139 100.

This shows that the situation has not changed at all. It can still be said that there are lacking points in the course books in our country. In another similar study by Peterson, Fennema and Carpenter (1989), it was seen that children were mostly presented results-unknown problems. As the course books include more results-unknown problems, these problems are mostly presented to children. Though this is the case, according to Greer (1997), carefully chosen word problems can provide a rich context for learning addition and subtraction concepts. The learning environment needs to be enriched especially by different types of problems (For example; enriching the textbooks with this perspective, increasing the teachers' awareness on this issue etc.). To conclude, the results of these studies are compatible with the studies by Saribaş & Aktaş Arnas, 2017; Olkun and Toluk, 2002; Peterson, Fennema and Carpenter, 1989; Despina & Harikleia, 2014; Pramjit and Teoh, 2010; Jeong and Lee; 2016) on the findings that all problems are not equally given importance in course books (change unknown problems, comparative problems) and this influence schildren's problem solving skills negatively.

At what level the students solve word problems based on addition and subtraction and the types of word problems which the students encounter in mathematics textbooks were investigated. The study showed that students had difficulties with initial-unknown addition and subtraction problems and comparative problems in comparison to other types. In addition to this, it was seen that word problems based on addition and subtraction mostly with result unknown were used whereas comparison problems were scarcely used in elementary school mathematics textbooks and workbooks that were designed by MoNE. When these results are considered, it is believed that presenting all types of problems in a balanced distribution in textbooks and workbooks will be beneficial. Besides, it is also important for teachers to use all types of problems in their classes so as to enable the students to experience and learn these problem situations.

Furthermore, the recommendations for further studies about this issue can be organised as: -This study was conducted with 3rd grade elementary school students. It can be replicated by involving all grades and comparative analysis can be made. -In this study, only the textbooks and workbooks which were assigned by MoNE were dealt with. As some of the teachers use different reference/source books, source books from different publishing companies can be included in future study. -In this study, the teachers' perspectives and the data about whether they use these types of word problems were not collected. In order to reveal the awareness levels of the teachers on this issue, a similar study can be carried out with elementary school teachers.

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2. Preschool Children's Skills in Solving Mathematical Word Problems. Educational Research and Reviews, 10(18), 2539-2549. Büyüköztürk, Ş. (2015).

3.The effect of instruction on children's solutions of addition and subtraction word problems.Educ. Stud. Math. 14(1), 55-72. Creswell, J. W. (2014).

§21. Theory about equation

Identify parts of an expression using mathematical terms (sum, term, product, factor, quotient, coefficient); view one or more parts of an expression as a single entity. For example, describe the expression 2 (8 + 7) as a product of two factors; view (8 + 7) as both a single entity and a sum of two terms.

Evaluate expressions at specific values of their variables. Include expressions that arise from formulas used in real-world problems. Perform arithmetic operations, including those involving whole-number exponents, in the conventional order when there are no parentheses to specify a particular order (Order of Operations). For example, use the formulas $V = s^3$ and $A = 6\ s^2$ to find the volume and surface area of a cube with sides of length s = 1/2.

Apply the properties of operations to generate equivalent expressions. For example, apply the distributive property to the expression 3 (2 + x) to produce the equivalent expression 6 + 3x; apply the distributive property to the expression 24x + 18y to produce the equivalent expression 6 (4x + 3y); apply properties of operations to y + y + y to produce the equivalent expression 3y.

Identify when two expressions are equivalent (i.e., when the two expressions name the same number regardless of which value is substituted into them). For example, the expressions y + y + y and 3y are equivalent because they name the same number regardless of which number y stands for..

Reason about and solve one-variable equations and inequalities.

Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

Solve real-world and mathematical problems by writing and solving equations of the form $x + p = q$ and $px = q$ for cases in which p, q and x are all nonnegative rational numbers.

Write an inequality of the form $x > c$ or $x < c$ to represent a constraint or condition in a real-world or mathematical problem. Recognize that inequalities of the form $x > c$ or $x < c$ have infinitely many solutions; represent solutions of such inequalities on number line diagrams.

Represent and analyze quantitative relationships between dependent and independent variables.

Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.

Expression of a variable. Algebraic expression.

An algebraic expression comprises both numbers and variables together with at least one arithmetic operation.

Example

$4\cdot x-34\cdot x-3$

A variable, as we learned in pre-algebra, is a letter that represents unspecified numbers. One may use a variable in the same manner as all other numerals:

Addition	$4+y$	4 plus y
Subtraction	$x-5$	x minus 5
	$8-a$	8 minus a
Division	$z/7$	z divided by 7
	$14/x$	14 divided by x
Multiplication	$9x$	9 times x

To evaluate an algebraic expression you have to substitute each variable with a number and perform the operations included.

Example

Evaluate the expression when $x=5$

$4\cdot x-34\cdot x-3$

First we substitute x with 5

$4\cdot 5-34\cdot 5-3$

And then we calculate the answer

$20-3=1720-3=17$

An expression that represents repeated multiplication of the same factor is called a power e.g.

$5\cdot 5\cdot 5=1255\cdot 5\cdot 5=125$

A power can also be written as

$53=12553=125$

Where 5 is called the base and 3 is called the exponent. The exponent corresponds to the number of times the base is used as a factor.

$53=5\cdot 5\cdot 553=5\cdot 5\cdot 5$

3^1	3 to the first power	3

4^2	4 to the second power or 4 squared	4·4
5^3	5 to the third power or 5 cubed	5·5·5
2^6	2 to the sixth power	2·2·2·2·2·2

Why do we have math if we can describe things in words?

Algebraic expressions are useful because they represent the value of an expression for all of the values a variable can take on.

Sometimes in math, we describe an expression with a phrase. For example, the phrase

"two more than five"

can be written as the expression

5+25+25, plus, 2.

Similarly, when we describe an expression in words that includes a variable, we're describing an algebraic expression, an expression with a variable.

For example,

"three more than xxx"

can be written as the algebraic expression

x+ 3 x, plus, 3.

But why? Why use math if we can describe things in words? One of the many reasons is that math is more precise and easier to work with than words are. This is a question you should keep thinking about as we dig deeper into algebra.

Different words for addition, subtraction, multiplication, and division

Here is a table that summarizes common words for each operation:

Operation	**Words**	**Example algebraic expression**
Addition	Plus, sum, more than, increased by	x + 3 x, plus, 3
Subtraction	Subtracted, minus, difference, less than, decreased by	p – 6 p, minus, 6
Multiplication	Times, product	8k 8, k times
Division	Divided, quotient	a \div 9 a÷9 a, divided by, 9

For example, the word product tells us to use multiplication. So, the phrase

"the product of eight and kkk"

can be written as

8k8k8, k.

Let's take a look at a trickier example

Write an expression for "mmm decreased by seven".

Notice that the phrase "decreased by" tells us to use subtraction.

So, the expression is m - 7m−7m, minus, 7.

Before you start solving equations, you should have a basic understanding of variables, as well as translating and evaluating algebraic expressions.

Variables

A variable is a letter used to stand for a number. The letters x, y, z, a, b, c, m, and n are probably the most commonly used variables. The letters e and i have special values in algebra and are usually not used as variables. The letter o is usually not used because it can be mistaken for 0 (zero).

Algebraic expressions

Variables are used to change verbal expressions into algebraic expressions, that is, expressions that are composed of letters that stand for numbers. Key words that can help you translate words into letters and numbers include:

- For addition: sum, more than, greater than, increase
- For subtraction: minus, less than, smaller than, decrease
- For multiplication: times, product, multiplied by, of
- For division: halve, divided by, ratio.

Example 1

Give the algebraic expression for each of the following.

1. the sum of a number and 5
2. the number minus 4
3. six times a number
4. x divided by 7
5. three more than the product of 2 and x

1. the sum of a number and 5: x + 5 or 5 + x
2. the number minus 4: x – 4
3. six times a number: 6 x
4. x divided by 7: $x/7$ or $\frac{x}{7}$
5. three more than the product of 2 and x: 2 x + 3

Evaluating expressions

To evaluate an expression, just replace the variables with grouping symbols, insert the values given for the variables, and do the arithmetic. Remember to follow the order of the operations: parentheses, exponents, multiplication/division, addition/subtraction.

Example 2

Evaluate each of the following.

1. x + 2 y if x = 2 and y = 5
2. a + bc – 3 if a = 4, b = 5, and c = 6
3. m ^{2} + 4 n + 1 if m = 3 and n = 2
4. $\frac{b+c}{7}+\frac{c}{a}$ if a = 2, b = 3, and c = 4
5. –5 xy + z if x = 6, y = 7, and z = 1

1.
$$\begin{aligned} x+2&=(2)+2(5)\\ &=2+10\\ &=12 \end{aligned}$$

2.
$$\begin{aligned} a+bc-3&=(4)+(5)(6)-3\\ &=4+30-3\\ &=34-3\\ &=31 \end{aligned}$$

3.
$$\begin{aligned} m^2+4n+1&=(3)^2+4(2)+1\\ &=9+8+1\\ &=17+1\\ &=18 \end{aligned}$$

4.
$$\begin{aligned} \frac{b+c}{7}+\frac{c}{a}&=\frac{(3)+(4)}{7}+\frac{(4)}{(2)}\\ &=\frac{7}{7}+2\\ &=1+2\\ &=3 \end{aligned}$$

5.
$$\begin{aligned} -5xy+z&=-5(6)(7)+(1)\\ &=-5(42)+1\\ &=-210+1\\ &=-209 \end{aligned}$$

§22. Theorems about equivalence equations. Linear equation and its solution

Equivalent equations are two **equations** set equal to each other so that the variable has the same value in each **equation**. **Equivalent equations** are algebraic **equations** that have identical solutions or roots. Adding or subtracting the same number or expression to both sides of an **equation** produces an **equivalent equation**. Multiplying or dividing both sides of an **equation** by the same non-zero number produces an **equivalent equation.**

The simplest examples of equivalent equations don't have any variables. For example, these three equations are equivalent to each other:

- $3 + 2 = 5$
- $4 + 1 = 5$
- $5 + 0 = 5$

Recognizing these equations are equivalent is great, but not particularly useful. Usually, an equivalent equation problem asks you to solve for a variable to see if it is the same (the same **root**) as the one in another equation.

For example, the following equations are equivalent:

- $x = 5$
- $-2x = -10$

In both cases, $x = 5$. How do we know this? How do you solve this for the "$-2x = -10$" equation? The first step is to know the rules of equivalent equations:

- Adding or subtracting the same number or expression to both sides of an equation produces an equivalent equation.

- Multiplying or dividing both sides of an equation by the same non-zero number produces an equivalent equation.
- Raising both sides of the equation to the same odd power or taking the same odd root will produce an equivalent equation.
- If both sides of an equation are non-negative, raising both sides of an equation to the same even power or taking the same even root will give an equivalent equation.

Example

Putting these rules into practice, determine whether these two equations are equivalent:

- $x + 2 = 7$
- $2x + 1 = 11$

To solve this, you need to find "x" for each equation. If "x" is the same for both equations, then they are equivalent. If "x" is different (i.e., the equations have different roots), then the equations are not equivalent. For the first equation:

- $x + 2 = 7$
- $x + 2 - 2 = 7 - 2$ (subtracting both sides by same number)
- $x = 5$

For the second equation:

- $2x + 1 = 11$
- $2x + 1 - 1 = 11 - 1$ (subtracting both sides by the same number)
- $2x = 10$
- $2x/2 = 10/2$ (dividing both sides of the equation by the same number)
- $x = 5$

So, yes, the two equations are equivalent because $x = 5$ in each case.

Practical Equivalent Equations

You can use equivalent equations in daily life. It's particularly helpful when shopping. For example, you like a particular shirt. One company offers the shirt for \$6 and has \$12 shipping, while another company offers the shirt for \$7.50 and has \$9 shipping. Which shirt has the best price? How many shirts (maybe you want to get them for friends) would you have to buy for the price to be the same for both companies?

To solve this problem, let "x" be the number of shirts. To start with, set x =1 for the purchase of one shirt. For company #1:

- Price = 6x + 12 = (6)(1) + 12 = 6 + 12 = \$18

For company #2:

- Price = 7.5x + 9 = (1)(7.5) + 9 = 7.5 + 9 = \$16.50

So, if you're buying one shirt, the second company offers a better deal.

To find the point where prices are equal, let "x" remain the number of shirts, but set the two equations equal to each other. Solve for "x" to find how many shirts you'd have to buy:

- 6x + 12 = 7.5x + 9
- 6x - 7.5x = 9 - 12 (subtracting the same numbers or expressions from each side)
- -1.5x = -3
- 1.5x = 3 (dividing both sides by the same number, -1)
- x = 3/1.5 (dividing both sides by 1.5)
- x = 2

If you buy two shirts, the price is the same, no matter where you get it. You can use the same math to determine which company gives you a better deal with larger orders and also to calculate how much you'll save using one company over the other. See, algebra is useful!

Equivalent Equations With Two Variables

If you have two equations and two unknowns (x and y), you can determine whether two sets of linear equations are equivalent.

For example, if you're given the equations:

- -3x + 12y = 15

- 7x - 10y = -2

You can determine whether the following system is equivalent:

- -x + 4y = 5
- 7x -10y = -2

To solve this problem, find "x" and "y" for each system of equations. If the values are the same, then the systems of equations are equivalent.

Start with the first set. To solve two equations with two variables, isolate one variable and plug its solution into the other equation. To isolate the "y" variable:

- -3x + 12y = 15
- -3x = 15 - 12y
- x = -(15 - 12y)/3 = -5 + 4y (plug in for "x" in the second equation)
- 7x - 10y = -2
- 7(-5 + 4y) - 10y = -2
- -35 + 28y - 10y = -2
- 18y = 33
- y = 33/18 = 11/6

Now, plug "y" back into either equation to solve for "x":

- 7x - 10y = -2
- 7x = -2 + 10(11/6)

Working through this, you'll eventually get x = 7/3.

To answer the question, you could apply the same principles to the second set of equations to solve for "x" and "y" to find that yes, they are indeed equivalent. It's easy to get bogged down in the algebra, so it's a good idea to check your work using an online equation solver.

However, the clever student will notice the two sets of equations are equivalent without doing any difficult calculations at all. The only difference between the first equation in each set is that the first one is three times the second one (equivalent). The second equation is exactly the same.

We consider a (nonhomogeneous) system of linear equations given by a matrix equation of the form

(1) $Ax=bAx=b$

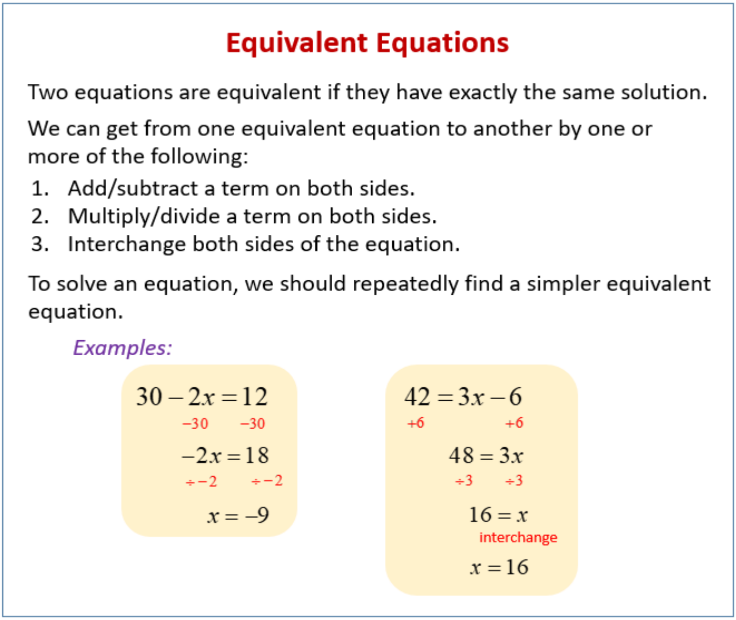
and the corresponding homogeneous system

(2) $Ax=0,Ax=0,$

where $A\in Mm\times n(K)A\in Mm\times n(K)$ and $b\in Km.b\in Km.$

A necessary and sufficient condition for any solution to exist provides. Examples, solutions, and videos that will help GMAT students review equivalent equations and equations with no solution.

The following diagram shows equivalent equations and how to use them. Scroll down the page for more examples and solutions for equivalent equations.



Equivalent Equations

Two equations are equivalent if they have exactly the same solution.

We can get from one equivalent equation to another by one or more of the following:

1. Add/subtract a term on both sides.
2. Multiply/divide a term on both sides.
3. Interchange both sides of the equation.

To solve an equation, we should repeatedly find a simpler equivalent equation.

Examples:

$30-2x=12$
$-30 \quad -30$
$-2x=18$
$\div -2 \quad \div -2$
$x=-9$

$42=3x-6$
$+6 \quad +6$
$48=3x$
$\div 3 \quad \div 3$
$16=x$
interchange
$x=16$

Equivalent Equations

Two equations are equivalent if they have the same solution. We can add, subtract, multiply, divide by the same number to each side of the equal sign and have equivalent equations.

§23. System of two-dimensional linear equations and methods of their solution

Systems of Linear Equations

A Linear Equation is an **equation** for a **line**.

A linear equation is not always in the form **$y = 3.5 - 0.5x$**,

It can also be like **$y = 0.5(7 - x)$**

Or like **$y + 0.5x = 3.5$**

Or like **$y + 0.5x - 3.5 = 0$** and more.

(Note: those are all the same linear equation!)

A **System** of Linear Equations is when we have **two or more linear equations** working together.

Example: Here are two linear equations:

$$\begin{array}{ccccc} 2x & + & y & = & 5 \\ -x & + & y & = & 2 \end{array}$$

Together they are a system of linear equations.

Can you discover the values of **x** and **y** yourself? (Just have a go, play with them a bit.)

Let's try to build and solve a real world example:

Example: You versus Horse

It's a race!

You can run **0.2 km** every minute.

The Horse can run **0.5 km** every minute. But it takes 6 minutes to saddle the horse.

How far can you get before the horse catches you?

We can make **two** equations (**d**=distance in km, **t**=time in minutes)

- You run at 0.2km every minute, so **d = 0.2t**
- The horse runs at 0.5 km per minute, but we take 6 off its time: **d = 0.5(t−6)**

So we have a **system** of equations (that are **linear**):

- **d = 0.2t**
- **d = 0.5(t−6)**

We can solve it on a graph:

Do you see how the horse starts at 6 minutes, but then runs faster?

It seems you get caught after 10 minutes ... you only got 2 km away.

Run faster next time.

So now you know what a System of Linear Equations is.

Let us continue to find out more about them ....

Solving

There can be many ways to solve linear equations!

Let us see another example:

Example: Solve these two equations:

- $x + y = 6$
- $-3x + y = 2$

The two equations are shown on this graph:

Our task is to find where the two lines cross.

Well, we can see where they cross, so it is already solved graphically.

But now let's solve it using Algebra!

Hmmm ... how to solve this? **There can be many ways!** In this case both equations have "y" so let's try subtracting the whole second equation from the first:

$x + y - (-3x + y) = 6 - 2$

Now let us simplify it:

$x + y + 3x - y = 6 - 2$

$4x = 4$

$x = 1$

So now we know the lines cross at **x=1**.

And we can find the matching value of **y** using either of the two original equations (because we know they have the same value at x=1). Let's use the first one (you can try the second one yourself):

$x + y = 6$

$1 + y = 6$

$y = 5$

And the solution is:

$x = 1$ and $y = 5$

And the graph shows us we are right!

Linear Equations

Only simple variables are allowed in linear equations. **No x^2, y^3, $\sqrt{x}$, etc**:

Dimensions

A **Linear Equation** can be in 2 dimensions ... (such as **x** and **y**)

... or in 3 dimensions ... (it makes a plane)

... or 4 dimensions ...

... or more!

Common Variables

For the equations to "work together" they share one or more variables:

A System of Equations has **two or more equations** in **one or more variables**

Many Variables

So a System of Equations could have **many** equations and **many** variables.

Example: 3 equations in 3 variables

$$\begin{array}{ccccccc} 2x & + & y & - & 2z & = & 3 \\ x & - & y & - & z & = & 0 \\ x & + & y & + & 3z & = & 12 \end{array}$$

There can be any combination:

- 2 equations in 3 variables,
- 6 equations in 4 variables,
- 9,000 equations in 567 variables,
- etc.

Solutions

When the number of equations is the **same** as the number of variables there is **likely** to be a solution. Not guaranteed, but likely. In fact there are only three possible cases:

- **No** solution
- **One** solution
- **Infinitely many** solutions

When there is **no solution** the equations are called **"inconsistent"**. **One** or **infinitely many solutions** are called **"consistent".** Here is a diagram for **2 equations in 2 variables**:

Independent

"Independent" means that each equation gives new information. Otherwise they are **"Dependent"**.

Also called "Linear Independence" and "Linear Dependence"

Example:

- $x + y = 3$
- $2x + 2y = 6$

Those equations are **"Dependent"**, because they are really the **same equation**, just multiplied by 2.

So the second equation gave **no new information**.

Where the Equations are True

The trick is to find where **all** equations are **true at the same time**.

True? What does that mean?

Example: You versus Horse

The "you" line is **true all along its length** (but nowhere else).

Anywhere on that line **d** is equal to **0.2t**

- at $t=5$ and $d=1$, the equation is **true** (Is $d = 0.2t$? Yes, as $\mathbf{1 = 0.2\times5}$ is true)

- at t=5 and d=3, the equation is **not** true (Is d = 0.2t? No, as **3 = 0.2×5 is not true**)

Likewise the "horse" line is also **true all along its length** (but nowhere else).

But only at the point where they **cross** (at t=10, d=2) are they **both true**.

So they have to be true **simultaneously** ...

... that is why some people call them **"Simultaneous Linear Equations"**

Solve Using Algebra

It is common to use Algebra to solve them.

Here is the "Horse" example solved using Algebra:

Example: You versus Horse

The system of equations is:

- $d = 0.2t$
- $d = 0.5(t-6)$

In this case it seems easiest to set them equal to each other:

$d = 0.2t = 0.5(t-6)$

Start with:$0.2t = 0.5(t - 6)$

Expand **$0.5(t-6)$**:$0.2t = 0.5t - 3$

Subtract **0.5t** from both sides:$-0.3t = -3$

Divide both sides by **−0.3**:$t = -3/-0.3 =$ **10** minutes

Now we know **when** you get caught!

Knowing **t** we can calculate **d**:$d = 0.2t = 0.2\times10 =$ **2** km

And our solution is:

$t = 10$ minutes and $d = 2$ k

Algebra vs Graphs

Why use Algebra when graphs are so easy? Because:

More than 2 variables can't be solved by a simple graph.

So Algebra comes to the rescue with two popular methods:

- Solving By Substitution
- Solving By Elimination

We will see each one, with examples in 2 variables, and in 3 variables. Here goes ...

Solving By Substitution

These are the steps:

- Write one of the equations so it is in the style **"variable = ..."**
- **Replace** (i.e. substitute) that variable in the other equation(s).
- **Solve** the other equation(s)
- (Repeat as necessary)

Here is an example with **2 equations in 2 variables**:

Example:

- $3x + 2y = 19$
- $x + y = 8$

We can start with **any equation** and **any variable**.

Let's use the second equation and the variable "y" (it looks the simplest equation).

Write one of the equations so it is in the style "variable = ...":

We can subtract x from both sides of $x + y = 8$ to get $\mathbf{y = 8 - x}$. Now our equations look like this:

- $3x + 2y = 19$
- $\mathbf{y = 8 - x}$

Now replace "y" with "8 − x" in the other equation:

- $3x + 2\mathbf{(8 - x)} = 19$
- $y = 8 - x$

Solve using the usual algebra methods:

Expand **2(8−x)**:

- 3x + **16 − 2x** = 19
- y = 8 − x

Then **3x−2x = x**:

- **x** + 16 = 19
- y = 8 − x

And lastly **19−16=3**

- **x = 3**
- y = 8 − x

Now we know what **x** is, we can put it in the **y = 8 − x** equation:

- x = 3
- y = 8 **− 3** = 5

And the answer is:

x = 3
y = 5

Note: because there **is** a solution the equations are **"consistent"**

Check: why don't you check to see if x = 3 and y = 5 works in both equations?

Solving By Substitution: 3 equations in 3 variables

OK! Let's move to a **longer** example: **3 equations in 3 variables**.

This is **not hard** to do... it just takes a **long time**!

Example:

- x + z = 6
- z − 3y = 7
- 2x + y + 3z = 15

We should line up the variables neatly, or we may lose track of what we are doing:

x + z = 6

− 3y + z = 7

2x + y + 3z = 15

WeI can start with any equation and any variable. Let's use the first equation and the variable "x".

Write one of the equations so it is in the style "variable = ...":

x **=** **6 − z**

− 3y + z = 7

2x + y + 3z = 15

Now replace "x" with "6 − z" in the other equations:

(Luckily there is only one other equation with x in it)

x = 6 − z

− 3y + z = 7

2**(6−z)** + y + 3z = 15

Solve using the usual algebra methods:

2(6−z) + y + 3z = 15 simplifies to **y + z = 3**:

x					=	6 − z	
	−	3y	+	z	=	7	
		y	+	z	=	3	

Good. We have made some progress, but not there yet.

Now **repeat the process**, but just for the last 2 equations.

Write one of the equations so it is in the style "variable = ...":

Let's choose the last equation and the variable z:

x					=	6 − z	
	−	3y	+	z	=	7	
				z	**=**	**3 − y**	

Now replace "z" with "3 − y" in the other equation:

x					=	6 – z	
	–	3y	+	**3 − y**	=	7	
				z	=	3 – y	

Solve using the usual algebra methods:

−3y + (3−y) = 7 simplifies to **−4y = 4**, or in other words **y = −1**

x					=	6 − z	
		y			**=**	**−1**	
				z	=	3 − y	

Almost Done!

Knowing that **y = −1** we can calculate that **z = 3−y = 4**:

x					=	6 − z	
		y			=	−1	
				z	**=**	**4**	

And knowing that **z = 4** we can calculate that **x = 6−z = 2**:

x					**=**	**2**
		y			=	−1
				z	=	4

And the answer is:

x = 2

y = −1

z = 4

Check: please check this yourself.

We can use this method for 4 or more equations and variables... just do the same steps again and again until it is solved.

Conclusion: Substitution works nicely, but does take a long time to do.

Solving By Elimination

Elimination can be faster ... but needs to be kept neat.

"Eliminate" means to **remove**: this method works by removing variables until there is just one left.

The idea is that we **can safely**:

- **multiply** an equation by a constant (except zero),
- **add** (or subtract) an equation on to another equation

Like in these examples:

WHY can we add equations to each other?

Imagine two really simple equations:

$$x - 5 = 3$$

$$5 = 5$$

We can add the "5 = 5" to "x − 5 = 3":

$$x - 5 + 5 = 3 + 5$$

$$x = 8$$

Try that yourself but use 5 = 3+2 as the 2nd equation

It will still work just fine, because both sides are equal (that is what the = is for!)

We can also swap equations around, so the 1st could become the 2nd, etc, if that helps.

OK, time for a full example. Let's use the **2 equations in 2 variables** example from before:

Example:

- $3x + 2y = 19$
- $x + y = 8$

Very important to keep things neat:

$$3x \quad + \quad 2y \quad = \quad 19$$

$$X \quad + \quad y \quad = \quad 8$$

Now ... our aim is to **eliminate** a variable from an equation.

First we see there is a "2y" and a "y", so let's work on that.

Multiply the second equation by 2:

3x	+	2y	=	19			
2x	+	**2**y	=	**16**			

Subtract the second equation from the first equation:

x			**=**	**3**
2x	+	2y	=	16

Yay! Now we know what x is!

Next we see the 2nd equation has "2x", so let's halve it, and then subtract "x":

Multiply the second equation by ½ (i.e. divide by 2):

x			=	3
x	**+**	**y**	**=**	**8**

Subtract the first equation from the second equation:

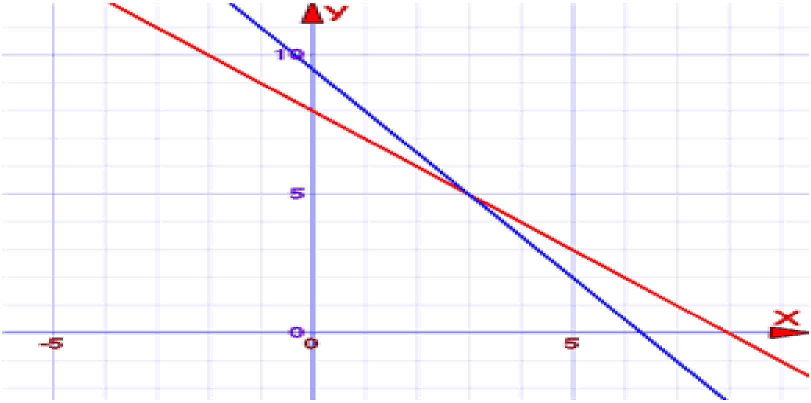
x			=	3
		y	**=**	**5**

Done!

And the answer is:

x = 3 and y = 5

And here is the graph:



The blue line is where **3x + 2y = 19** is true

The red line is where **x + y = 8** is true

At x=3, y=5 (where the lines cross) they are **both** true. **That** is the answer.

Here is another example:

Example:

- $2x - y = 4$
- $6x - 3y = 3$

Lay it out neatly:

2x	−	y	=	4
6x	−	3y	=	3

Multiply the first equation by 3:

6x	−	**3y**	**=**	**12**
6x	−	3y	=	3

Subtract the second equation from the first equation:

0	−	**0**	**=**	**9**
6x	−	3y	=	3

0 − 0 = 9 ???

What is going on here?

Quite simply, there is no solution.

They are actually parallel lines:

And lastly:

Example:

- 2x − y = 4
- 6x − 3y = 12

Neatly:

2x	−	y	=	4
6x	−	3y	=	12

Multiply the first equation by 3:

6x	−	**3y**	**=**	**12**
6x	−	3y	=	12

Subtract the second equation from the first equation:

0	−	**0**	**=**	**0**
6x	−	3y	=	3

0 − 0 = 0

Well, that is actually TRUE! Zero does equal zero ...

... that is because they are really the same equation ...

... so there are an Infinite Number of Solutions

They are the same line:

And so now we have seen an example of each of the three possible cases:

- **No** solution
- **One** solution

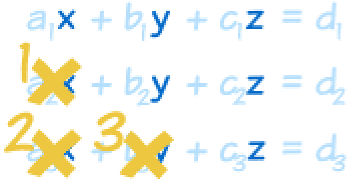
- **Infinitely many** solutions

Solving By Elimination: 3 equations in 3 variables

Before we start on the next example, let's look at an improved way to do things. Follow this method and we are less likely to make a mistake. First of all, eliminate the variables **in order**:

- Eliminate **x**s first (from equation 2 and 3, in order)
- then eliminate **y** (from equation 3)

So this is how we eliminate them:

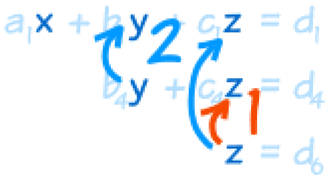


We then have this "triangle shape":

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_4y + c_4z &= d_4 \\ c_5z &= d_5 \end{aligned}$$

Now start at the bottom and **work back up** (called "Back-Substitution")

(put in **z** to find **y**, then **z** and **y** to find **x**):



And we are solved:

$$\begin{array}{lllll} x & & & = & d_8 \\ & y & & = & d_7 \\ & & z & = & d_6 \end{array}$$

ALSO, we will find it is easier to do **some** of the calculations in our head, or on scratch paper, rather than always working within the set of equations:

Example:

- $x + y + z = 6$
- $2y + 5z = -4$
- $2x + 5y - z = 27$

Written neatly:

$$\begin{array}{lllllll} x & + & y & + & z & = & 6 \\ & & 2y & + & 5z & = & -4 \\ 2x & + & 5y & - & z & = & 27 \end{array}$$

First, eliminate **x** from 2nd and 3rd equation.

There is no x in the 2nd equation ... move on to the 3rd equation:

Subtract 2 times the 1st equation from the 3rd equation (just do this in your head or on scratch paper):

$$\begin{array}{rl} 3^{rd}: & 2x + 5y - z = 27 \\ \text{subtract } 2 \times 1^{st}: & \underline{2x + 2y + 2z = 12} \\ & 0 \quad 3y - 3z = 15 \end{array}$$

And we get:

x	+	y	+	z	=	6
		2y	+	5z	=	−4
		3y	−	**3z**	**=**	**15**

Next, eliminate **y** from 3rd equation.

We **could** subtract 1½ times the 2nd equation from the 3rd equation (because 1½ times 2 is 3) ...

... but we can **avoid fractions** if we:

- multiply the 3rd equation by **2** and
- multiply the 2nd equation by **3**

and then do the subtraction ... like this:

$$\begin{array}{rl} 2\times 3^{rd}: & 6y-6z=30 \\ \text{Subtract } 3\times 2^{nd}: & \underline{6y+15z=-12} \\ & 0-21z=42 \\ & \quad\rightarrow z=-2 \end{array}$$

And we end up with:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 6 \\ & & 2y & + & 5z & = & -4 \\ & & & & \mathbf{z} & \mathbf{=} & \mathbf{-2} \end{array}$$

We now have that "triangle shape"!

Now go back up again "back-substituting":

We know **z**, so **2y+5z=−4** becomes **2y−10=−4**, then **2y=6**, so **y=3**:

x	+	y	+	z	=	6			
		y			**=**	**3**			
				z	=	−2			

Then **x+y+z=6** becomes **x+3−2=6**, so **x=6−3+2=5**

x					**=**	**5**
		y			=	3
				z	=	−2

And the answer is:

$$x = 5$$
$$y = 3$$
$$z = -2$$

Check: please check for yourself.

General Advice

Once you get used to the Elimination Method it becomes easier than Substitution, because you just follow the steps and the answers appear. But sometimes Substitution can give a quicker result.

- Substitution is often easier for small cases (like 2 equations, or sometimes 3 equations)
- Elimination is easier for larger cases

And it always pays to look over the equations first, to see if there is an easy shortcut ... so experience helps.

§24. The concept of inequality. Inequality - as a predicate

The concept of inequality

Inequality - as a predicate

As well as the familiar equals sign (=) it is also very useful to show if something is not equal to (≠) greater than (>) or less than (<)

These are the important signs to know:

=	When two values are equal we use the "equals" sign	example: **2+2 = 4**
≠	When two values are definitely **not** equal we use the "not equal to" sign	example: **2+2 ≠ 9**
<	When one value is smaller than another we use a "less than" sign	example: **3 < 5**

>	When one value is bigger than another we use a "greater than" sign	example: **9 > 6**

Less Than and Greater Than

The "less than" sign and the "greater than" sign look like a "V" on its side, don't they?

To remember which way around the "<" and ">" signs go, just remember:

- BIG > small
- small < BIG

The "small" end always points to the smaller number, like this:

Greater Than Symbol: **BIG > small**

Example:

10 > 5

"10 is **greater than** 5"

Or the other way around:

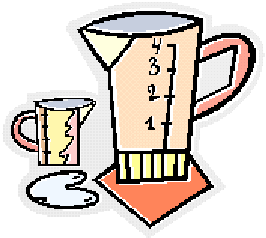
5 < 10

"5 is **less than** 10"

Do you see how the symbol "points at" the smaller value?

... Or Equal To ...

Sometimes we know a value is smaller, but **may also be equal to**!



Example, a jug can hold up to 4 cups of water.

So how much water is in it?

It could be 4 cups or it could be less than 4 cups: So until we measure it, all we can say is "less than **or equal to**" 4 cups.

To show this, we add an extra line at the bottom of the "less than" or "greater than" symbol like this:

The "less than **or equal to**" sign:		≤
The "greater than **or equal to**" sign:		≥

All The Symbols

Here is a summary of all the symbols:

Symbol	Words	Example Use
=	equals	1 + 1 = 2
≠	not equal to	1 + 1 ≠ 1
>	greater than	5 > 2
<	less than	7 < 9
≥	greater than or equal to	marbles ≥ 1
≤	less than or equal to	dogs ≤ 3

Why Use Them?

Because there are things we **do not know exactly ...**

... but can still **say something about**.

So we have ways of saying what we **do** know (which may be useful!)

Example: John had 10 marbles, but lost some. How many has he now?

Answer: He must have **less than** 10:

Marbles < 10

If John still has some marbles we can also say he has **greater than zero** marbles:

Marbles > 0

But if we thought John **could have** lost **all** his marbles we would say

Marbles ≥ 0

In other words, the number of marbles is greater than **or equal to** zero.

Combining

We can sometimes say two (or more) things on the one line:

Example: Becky starts with $10, buys something and says "I got change, too". How much did she spend?

Answer: Something greater than $0 and less than $10 (but NOT $0 or $10):

"What Becky Spends" > $0

"What Becky Spends" < $10

This can be written down in just one line:

$0 < "What Becky Spends" < $10

That says that $0 is less than "What Becky Spends" (in other words "What Becky Spends" is greater than $0) and what Becky Spends is also less than $10.

Notice that ">" was flipped over to "<" when we put it before what Becky spends. Always make sure the **small end points to the small value**.

Changing Sides

We saw in that previous example that when we change sides we flipped the symbol as well.

This:	Becky Spends > $0	(Becky spends greater than $0)
is the same as this:	$0 < Becky Spends	($0 is less than what Becky spends)

Just make sure the small end points to the small value!

Here is another example using "≥" and "≤":

Example: Becky has $10 and she is going shopping. How much will she spend **(without using credit)?**

Answer: Something greater than, or possibly equal to, $0 and less than, or possibly equal to, $10:

Becky Spends ≥ $0

Becky Spends ≤ $10

This can be written down in just one line:

$0 ≤ Becky Spends ≤ $10

Long Example: Cutting Rope

Here is an interesting example I thought of:

Example: Sam cuts a 10m rope into two. How long is the longer piece? How long is the shorter piece?

Answer: Let us call the **longer** length of rope "**L**", and the **shorter** length "**S**"

L must be greater than 0m (otherwise it isn't a piece of rope), and also less than 10m:

$$L > 0$$

$$L < 10$$

So:

$$0 < L < 10$$

That says that **L** (the Longer length of rope) is between 0 and 10 (but not 0 or 10)

The same thing can be said about the shorter length "**S**":

$$\mathbf{0 < S < 10}$$

But I did say there was a "shorter" and "longer" length, so we also know:

$$\mathbf{S < L}$$

(Do you see how neat mathematics is? Instead of saying "the shorter length is less than the longer length", we can just write "**S < L**")

We can combine all of that like this:

$$\mathbf{0 < S < L < 10}$$

That says a lot:

0 is less that the short length, the short length is less than the long length, the long length is less than 10.

Reading "backwards" we can also see:

10 is greater than the long length, the long length is greater than the short length, the short length is greater than 0.

It also lets us see that "S" is less than 10 (by "jumping over" the "L"), and even that 0<10 (which we know anyway), all in one statement.

NOW, I have one more trick. If Sam tried really hard he might be able to cut the rope EXACTLY in half, so each half is 5m, but we know he didn't because we said there was a "shorter" and "longer" length, so we also know:

$$S<5$$

and

$$L>5$$

We can put that into our very neat statement here:

$$0 < S < 5 < L < 10$$

And IF we thought the two lengths MIGHT be exactly 5 we could change that to

$$0 < S \le 5 \le L < 10$$

An Example Using Algebra

OK, this example may be complicated if you don't know Algebra, but I thought you might like to see it anyway:

Example: What is $x+3$, when we know that x is greater than 11?

If $x > 11$, then $x+3 > 14$

(Imagine that "x" is the number of people at your party. If there are more than 11 people at your party, and 3 more arrive, then there must be more than 14 people at your party now.)

§25. Numerical expression

Numerical Expression Definition

You have likely worked with numbers since your preschool days. You started counting numbers, and pretty soon you were adding them using your fingers, Cheerios, or any other objects you used to help you count. Since then, you are now (hopefully) able to look at a piece of paper or computer screen and add, subtract, multiply, and divide small numbers in your head.

When you look at a problem with numbers, you are most likely looking at a numerical expression. A **numerical expression** is a mathematical sentence involving only numbers and one or more operation symbols. Examples of operation symbols are the ones for addition, subtraction, multiplication, and division. They can also be the radical symbol (the square root symbol) or the absolute value symbol.

Examples of Numerical Expressions

The only requirements for a numerical expression are that it only contain numbers and operation symbols. Some numerical expressions have only one operation symbol. Others have two or more. Here are some examples of numerical expressions:

4 + 5

134 - 75

56 * 4 + 6

68 / 8 * 7 - 2 + 1

Examples of Non-Numerical Expressions

Because numerical expressions can only contain numbers, expressions containing **variables** (such as x or y) cannot be considered numerical expressions. They are actually called **algebraic expressions** instead. Here are two examples of algebraic expressions:

4x + 5

134 - x

Writing Numerical Expressions

When given a verbal or written-down **word problem**, it is important to be able to translate the words to a numerical expression so you can solve the problem. Here are a few examples.

Example 1: Amanda had 12 gummy bears and gave 2 to her brother. She then ate 4 gummy bears. How many gummy bears did Amanda have left?

Amanda started with 12, gave two away (-2) and ate 4 (-4). In this example, only the subtraction operation needed to be used. The numerical expression could be written as:

12 - 2 – 4

§26. Basic geometric concepts

The fundamental geometrical concepts depend on three basic concepts — point, line and plane. The terms cannot be precisely defined. However, the meanings of these terms are explained through examples.

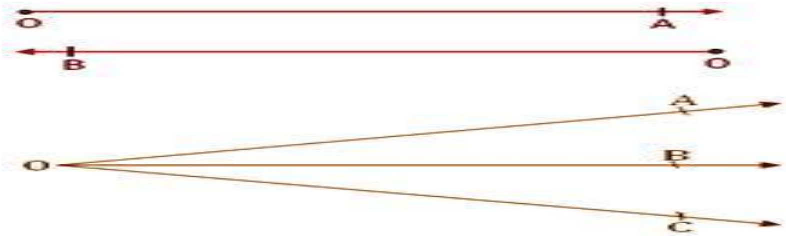
Point:

- It is the mark of position and has an exact location.
- It has no length, breadth or thickness.
- It is denoted by a dot made by the tip of a sharp pencil.
- It is denoted by capital letter.
- In the given figure P, Q, R represents different points.

Line:

- It is a straight path which can be extended indefinitely in both the directions.
- It is shown by two arrowheads in opposite directions.
- It does not have any fixed length.
- It has no endpoints.
- It is denoted as AB ↔ or BA↔ and is read as line AB or line in the BA.
- It can never be measured.
- Infinite number of points lie on the line.
- Sometimes it is also denoted by small letters of the English alphabet.

Ray:



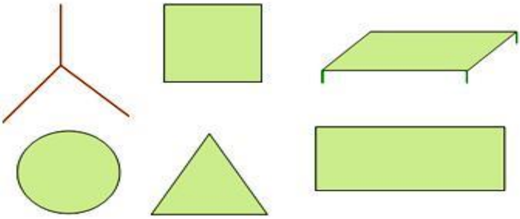
- It is a straight path which can be extended indefinitely in one direction only and the other end is fixed.
- It has no fixed length.
- It has one endpoint called the initial point.
- It cannot be measured.
- It is denoted as OA<span style='font-size: 50%'>→ and is read as ray OA.
- A number of rays can be drawn from an initial point O.
- Ray OA and ray OB are different because they are extended in different directions.
- Infinite points lie on the ray.

Line Segment:

1.It is a straight path which has a definite length.

2. It has two endpoints.
3. It is a part of the line.
4. It is denoted as AB or BA.
5. It is read as line segment AB or line segment BA.
6. The distance between A and B is called the length of AB.
7. Infinite number of points lies on a line segment.
8. Two line segments are said to be equal if they have the same length.

Plane:

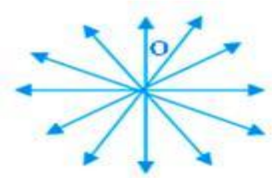


A smooth, flat surface gives us an idea of a plane. The surface of the table, wall, blackboard, etc., is smooth and flat. It extends endlessly in all the directions. It has no length, breadth or thickness. Here, we have shown a portion of a particular plane. We can draw certain figures like square, rectangle, triangle, and circle on the plane.

Hence, these figures are also called plane figures.

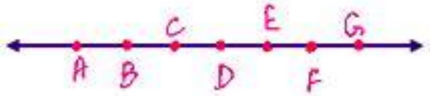
Incidence Properties of Lines in a Plane:

An infinite number of many lines can be drawn to pass through a given point in a plane. Through a given point in a plane, infinitely many lines can be drawn to pass through.

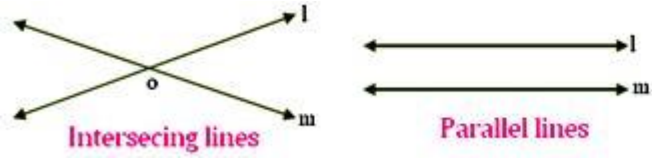


Two distinct points in a plane determine a unique line. One and only one line can be drawn to pass through two given points, i.e., two distinct points in a plane. This line lies wholly in the plane.

Infinite number of points lie on the line in a plane.



Two lines in a plane either intersect at a point or they are parallel to each other.



Collinear Points:

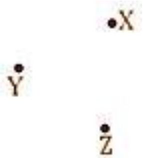
Two or more points which lie on the same line in a plane are called collinear points.

- The line is called the line of colinearity.
- Two points are always collinear.
- In the adjoining figure………

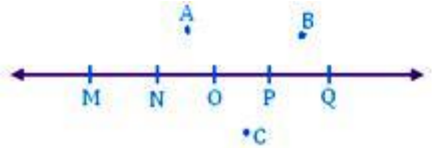
Points A, B, C are collinear lying on line

Points X, Y, Z are not collinear because all the three points do not lie on a line.

Hence, they are called non-collinear points.



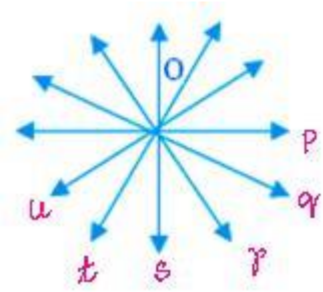
Similarly, here points M, N, O, P, Q are collinear points and A, B, C are non-collinear points.



Note:

Two points are always collinear.

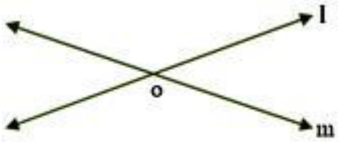
Concurrent Lines:



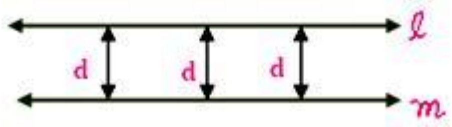
Three or more lines which pass through the same point are called concurrent lines and this common point is called the point of concurrence. In the adjoining figure, lines p, q, r, s, t, u intersect at point O and are called concurrent lines.

Two lines in a Plane:

Intersecting Lines: Two lines in a plane which cut each other at common point are called intersecting lines and the point is called the point of intersection. In the adjoining figure, lines l and m intersect at point O.



Parallel Lines: Two lines in a plane which do not intersect at any point, i.e., they do not have any point in common are called parallel lines. The distance between the two parallel lines remains the same throughout.



These are the fundamental geometrical concepts explained above using figures.

Line, angle, polygon and their definitions

When you start studying geometry, it is important to know and understand some basic concepts.

This page will help you understand the concept of dimensions in geometry, and work out whether you are working in one, two or three dimensions.

It also explains some of the basic terminology, and points you to other pages for more information.

Points, lines, and planes.

Other pages in this series explain about angles and shapes, including polygons, circles and other curved shapes, and three-dimensional shapes.ionary, 1989 edition

Geometry comes from the Greek meaning ‘earth measurement’ and is the visual study of shapes, sizes and patterns, and how they fit together in space. You will find that our geometry pages contain lots of diagrams to help you understand the subject.

When you’re faced with a problem involving geometry, it can be very helpful to draw yourself a diagram.

Working in Different Dimensions

No, not the space-time continuum! We’re talking about shapes that are in one, two and three dimensions. That is, objects that have length (one dimension), length and width (two dimensions) and length, width and depth or height (three dimensions).

Points: A Special Case: No Dimensions

A point is a single location in space. It is often represented by a dot on the page, but actually has no real size or shape.

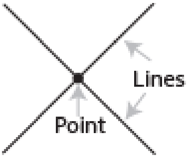
You cannot describe a point in terms of length, width or height, so it is therefore **non-dimensional**. However, a point may be described by co-ordinates. Co-ordinates do not define anything about the point other than its position in space, in relation to a reference point of known co-ordinates. You will come across point co-ordinates in many applications, such as when you are drawing graphs, or reading maps.

Almost everything in geometry starts with a point, whether it's a line, or a complicated three-dimensional shape.

Lines: One Dimension

A line is the shortest distance between two points. It has length, but no width, which makes it one-dimensional.

Wherever two or more lines meet, or intersect, there is a point, and the two lines are said to share a point:



Line segments and rays

There are two kinds of lines: those that have a defined start- and endpoint and those that go on for ever.

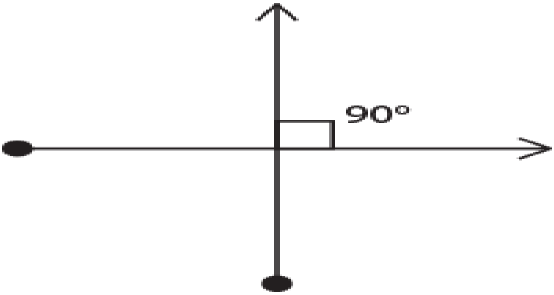
Lines that move between two points are called **line segments**. They start at a specific point, and go to another, the endpoint. They are drawn as a line between two points, as you would probably expect.

The second type of line is called a **ray**, and these go on forever. They are often drawn as a line starting from a point with an arrow on the other end:

Parallel and perpendicular lines

There are two types of lines that are particularly interesting and/or useful in mathematics. **Parallel lines** never meet or intersect. They simply go on forever side by side, a bit like railway lines. The convention for showing that lines are parallel in a diagram is to add ‘feathers’, which look like arrow heads.

Perpendicular lines intersect at a right angle, 90°:



Planes and Two-dimensional Shapes

Now that we have dealt with one dimension, it’s time to move into two.

A **plane** is a flat surface, also known as two-dimensional. It is technically unbounded, which means that it goes on for ever in any given direction and as such is impossible to draw on a page.

One of the key elements in geometry is how many dimensions you’re working in at any given time. If you are working in a single plane, then it’s either one (length) or two (length and width). With more than one plane, it must be three-dimensional, because height/depth is also involved.

Two-dimensional shapes include polygons such as squares, rectangles and triangles, which have straight lines and a point at each corner.

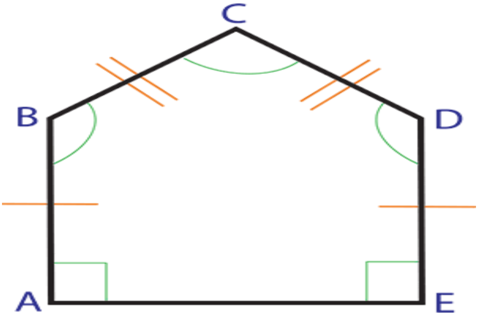
There is more about polygons in our page on **Polygons**. Other two-dimensional shapes include circles, and any other shape that includes a curve. You can find out more about these on our page, **Curved Shapes**.

Three Dimensions: Polyhedrons and Curved Shapes

Finally, there are also **three-dimensional shapes**, such as cubes, spheres, pyramids and cylinders.

To learn more about these see our page on **Three-Dimensional Shapes**.

Signs, Symbols and Terminology



The shape illustrated here is an irregular pentagon, a five-sided polygon with different internal angles and line lengths (see our page on **Polygons** for more about these shapes).

Degrees °are a measure of rotation, and define the size of the angle between two sides.

Angles are commonly marked in geometry using a segment of a circle (an arc), unless they are a right angle when they are ‘squared off'. Angle marks are indicated in green in the example here. See our page on **Angles** for more information.

Tick marks (shown in orange) indicate sides of a shape that have equal length (sides of a shape that are **congruent** or that match). The single lines show that the two vertical lines are the same length while the double lines show that the two diagonal lines are the same length. The bottom, horizontal, line in this example is a different length to the other 4 lines and therefore not marked. Tick marks can also be called '**hatch marks**'.

A vertex is the point where lines meet (lines are also referred to as rays or edges). The plural of vertex is vertices. In the example there are five vertices labelled A, B, C, D and E. Naming vertices with letters is common in geometry.

The angle symbol '∠' is used as a shorthand symbol in geometry when describing an angle. The expression **∠ABC** is shorthand to describe the angle between points A and C at point B. The middle letter in such expressions is always the vertex of the angle you are describing - the order of the sides is not important. **∠ABC** is the same as **∠CBA,** and both describe the vertex **B** in this example.

If you want to write the measured angle at point B in shorthand then you would use:

m∠ABC = 128° (m simply means 'measure')

or

m∠CBA = 128°

In our example we can also say:

m∠EAB=90°

m∠BCD=104°

Why Do These Concepts Matter?

Points, lines and planes underpin almost every other concept in geometry. Angles are formed between two lines starting from a shared point. Shapes, whether two-dimensional or three-dimensional, consist of lines which connect up points. Planes are important because two-dimensional shapes have only one plane; three-dimensional ones have two or more.

In other words, you really need to understand the ideas on this page before you can move on to any other area of geometry.

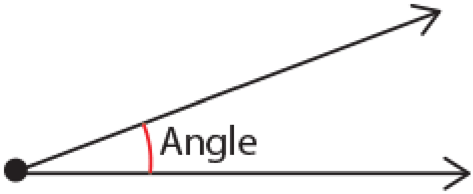
Introduction to Angles

Once you have mastered the idea of points, lines and planes, the next thing to consider is what happens when two lines or rays meet at a point, creating an angle between them.

Angles are used throughout geometry, to describe shapes such as polygons and polyhedrons, and to explain the behaviour of lines, so it's a good idea to become familiar with some of the terminology, and how we measure and describe angles.

What is an Angle?

Angles are formed between two rays extending from a single point:

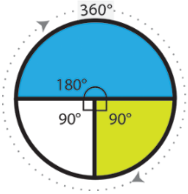


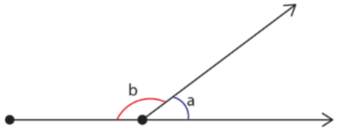
Angles are commonly drawn as an arc (part of a circle), as above.

Properties of Angles

Angles are measured in degrees, which is a measure of circularity, or rotation.

A full rotation, which would bring you back to face in the same direction, is 360°. A half-circle is therefore 180°, and a quarter-circle, or right angle, is 90°.





Two or more angles on a straight line add up to 180°. In the diagram above, the circle to the left is split into three sectors the angles of the green and white sectors are both 90°, adding up to 180°.

The figure to the right shows that angles a and b also add up to 180°. When you look at the diagram like this, it's easy to see this, but it's also surprisingly easy to forget in practice.

Naming Different Angles

An angle less than 90° is said to be acute, and one greater than 90° but less than 180° is obtuse.

An angle of exactly 180° is said to be straight. Angles greater than 180° are called reflex angles.

Different angles can be demonstrated on a clock face. The hour hand of the clock moves round as time passes through the day. The angle of the rotation is highlighted in green.

Acute angle
(less than 90°)

Right angle
(exactly 90°)

Obtuse angle
(greater than 90°
and less than 180°)

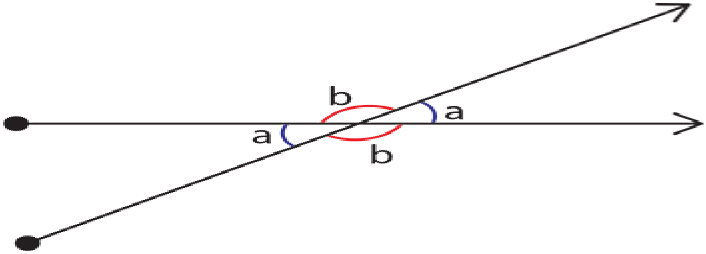
Straight angle
(exactly 180°)

Reflex angle
(greater than 180°)

Complete rotation
(360°)

Opposite Angles: Intersecting Lines

When two lines intersect, the opposite angles are equal. In this case, not only are a and a the same, but, of course, a and b add up to 180°:

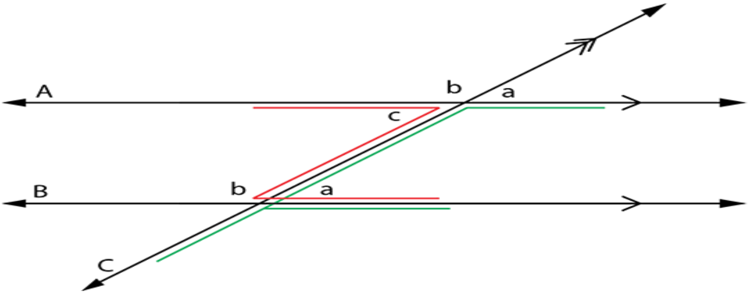


Intersections with parallel lines: a bit of a special case

Our page An Introduction to Geometry introduces the concept of parallel lines: lines that go on forever side by side and never cross, like railway lines.

The angles around any lines intersecting parallel lines also have some interesting properties.

If two parallel lines (A and B) are intersected by a third straight line (C), then the angle at which the intersecting line crosses will be the same for both parallel lines.

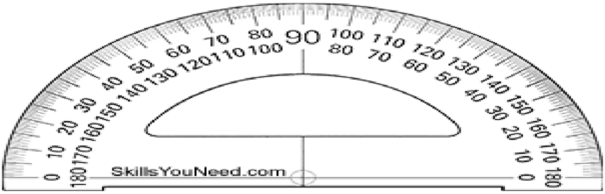


The two angles a and the two angles b are said to be **corresponding.**

You will also immediately see that a and b add up to 180°, since they are on a straight line.

Angle c, which you will realise from the previous section is identical to a, is said to be **alternate** with a.ndngles

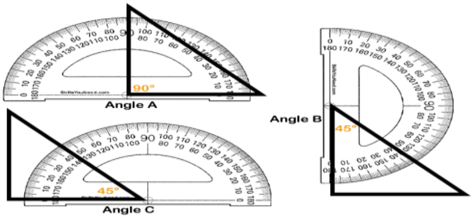
Measuring Angles



A protractor is commonly used to measure angles. Protractors are usually circular or semi-circular and made of transparent plastic, so that they can be placed over shapes drawn on a piece of paper, allowing you to take a measurement of the angle.

This example demonstrates how to use a protractor to measure the three angles of a triangle, but the same method applies to other shapes or any angles that you want to measure.

- Line up the central mark on the base of your protractor with the **vertex, or point at which** the lines meet. The triangle has three vertices, one for each angle to measure.
- Most protractors have a bi-directional scale meaning that you can take a measurement in either direction. Make sure you use the correct scale – you should be able to tell easily if your angle is greater than or less than 90° and therefore use the right scale. If you're not sure, take a quick look back to our section on naming angles.



In this example the recorded angles are A=90° B=45° and C=45°.

Polygons are often defined by their internal angles, and the total of the internal angles depends on the number of sides. For example, the internal angles of a triangle always add up to 180°. For more about this, see our page on **Polygons**. sphere (and there is more about this in our page on **Curved Shapes**).

Moving on…

Once you understand about angles, and how to measure them, you can put this into practice with polygons and polyhedrons of all kinds, and also use your knowledge to calculate area (there is more about this in our page on Calculating Area).

Reference

Read more at: https://www.skillsyouneed.com/num/angles.html

§27. Properties of Polygons

This page examines the properties of two-dimensional or ‘plane’ polygons. A polygon is any shape made up of straight lines that can be drawn on a flat surface, like a piece of paper. Such shapes include squares, rectangles, triangles and pentagons but not circles or any other shape that includes a curve.

Understanding shapes is important in mathematics. You will certainly be required to learn about shapes at school but understanding the properties of shapes has many practical applications in professional and real-life situations too.

Many professionals need to understand the properties of shapes, including engineers, architects, artists, real-estate agents, farmers and construction workers.

You may well need to understand shapes when doing home improvements and DIY, when gardening and even when planning a party.

Number of Sides

Polygons are usually defined by the number of sides that they have.

Three-Sided PolygonsTriangles

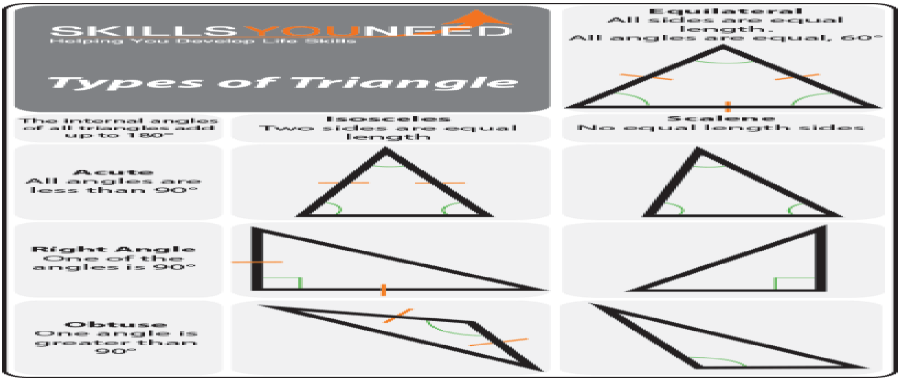
A three-sided polygon is a triangle. There are several different types of triangle (see diagram), including:

- **Equilateral** – all the sides are equal lengths, and all the internal angles are 60°.
- **Isosceles** – has two equal sides, with the third one a different length. Two of the internal angles are equal.
- **Scalene** – all three sides, and all three internal angles, are different.

Triangles can also be described in terms of their internal angles (see our page on **Angles** for more about naming angles). The internal angles of a triangle always add up to 180°.

A triangle with only **acute** internal angles is called an acute (or acute-angled) triangle. One with one **obtuse** angle and two acute angles is called obtuse (obtuse-angled), and one with a **right angle** is known as right-angled.

Each of these will also be either **equilateral, isosceles** or **scalene**.

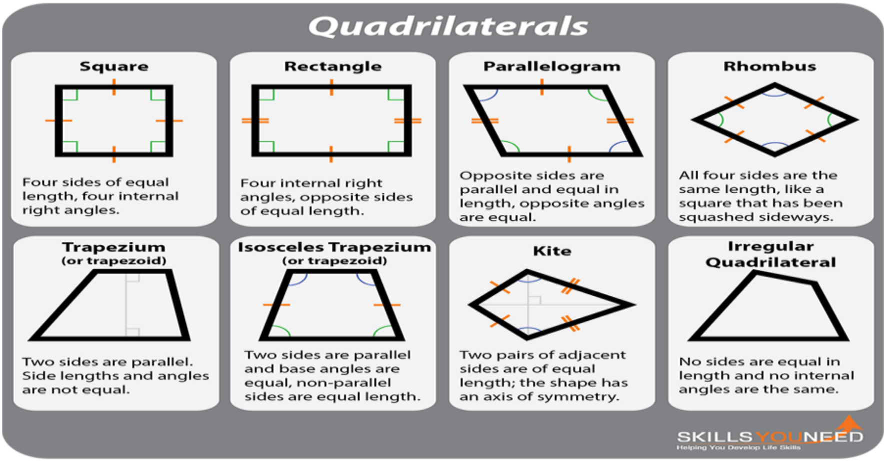


Four-Sided Polygons - Quadrilaterals

Four-sided polygons are usually referred to as quadrilaterals, quadrangles or sometimes tetragons. In geometry the term quadrilateral is commonly used. The term quadrangle is often used to describe a rectangular enclosed outdoor space, for example ‘the freshers assembled in the college quadrangle’. The term tetragon is consistent with polygon, pentagon etc. You may come across it occasionally, but it is not commonly used in practice.

The family of quadrilaterals includes the square, rectangle, rhombus and other parallelograms, trapezium/trapezoid and kite.

The internal angles of all quadrilaterals add up to 360°.



- **Square**: Four sides of equal length, four internal right angles.
- **Rectangle**: Four internal right angles, opposite sides of equal length.
- **Parallelogram**: Opposite sides are parallel, opposite sides are equal in length, opposite angles are equal.
- **Rhombus**: A special type of parallelogram in which all four sides are the same length, like a square that has been squashed sideways.
- **Trapezium (or trapezoid)**: Two sides are parallel, but the other two sides are not. Side lengths and angles are not equal.
- **Isosceles Trapezium (or trapezoid)**: Two sides are parallel and base angles are equal, meaning that non-parallel sides are also equal in length.
- **Kite**: Two pairs of adjacent sides are of equal length; the shape has an axis of symmetry.
- **Irregular Quadrilateral**: a four-sided shape where no sides are equal in length and no internal angles are the same. All internal angles still add up to 360°, as with all other regular quadrilaterals.

More than Four Sides

A five-sided shape is called a pentagon.

A six-sided shape is a hexagon, a seven-sided shape a heptagon, while an octagon has eight sides…

There are names for many different types of polygons, and usually the number of sides is more important than the name of the shape.

There are two main types of polygon - regular and irregular.

A **regular polygon** has equal length sides with equal angles between each side. Any other polygon is an **irregular polygon**, which by definition has unequal length sides and unequal angles between sides.

Circles and shapes that include curves are not polygons - a polygon, by definition, is made up of straight lines. See our pages on **circles and curved shapes** for more.

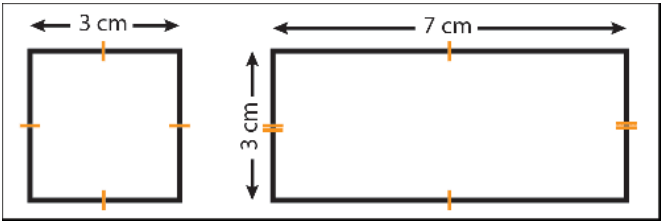
Angles Between Sides

The angles between the sides of shapes are important when defining and working with polygons. See our page on Angles for more about how to measure angles.

The Length of the Sides

As well as the number of sides and the angles between sides, the length of each side of shapes is also important.

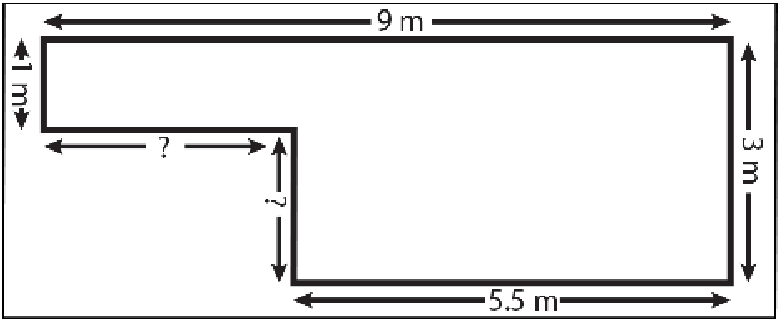
The length of the sides of a plane shape enables you to calculate the shape's **perimeter** (the distance around the outside of the shape) and **area** (the amount of space inside the shape).



If your shape is a regular polygon (such as a square in the example above) then it is only necessary to measure one side as, by definition, the other sides of a regular polygon are the same length. It is common to use tick marks to show that all sides are an equal length.

In the example of the rectangle we needed to measure two sides - the two unmeasured sides are equal to the two measured sides.

It is common for some dimensions not to be shown for more complex shapes. In such cases missing dimensions can be calculated.



In the example above, two lengths are missing.

The missing horizontal length can be calculated. Take the shorter horizontal known length from the longer horizontal known length.

$$9m - 5.5m = 3.5m.$$

The same principle can be used to work out the missing vertical length. That is:

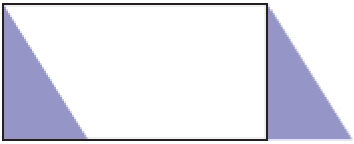
$$3m - 1m = 2m.$$

Bringing All the Information Together: Calculating the Area of Polygons

The simplest and most basic polygon for the purposes of calculating area is the quadrilateral. To obtain the area, you simply multiple length by vertical height.

For parallelograms, note that vertical height is **NOT** the length of the sloping side, but the vertical distance between the two horizontal lines.

This is because a parallelogram is essentially a rectangle with a triangle cut off one end and pasted onto the other:

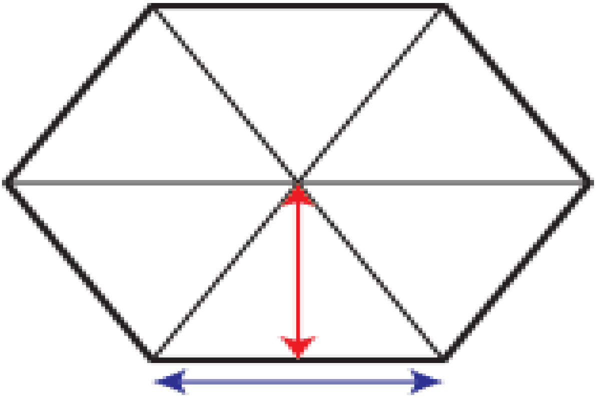


You can see that if you remove the left-hand blue triangle, and stick it onto the other end, the rectangle becomes a parallelogram.

The area is length (the top horizontal line) multiplied by height, the vertical distance between the two horizontal lines.

To work out the area of a **triangle**, you multiple length by vertical height (that is, the vertical height from the bottom line to the top point), and halve it. This is essentially because a triangle is half a rectangle.

To calculate the area of any regular polygon, the easiest way is to divide it into triangles, and use the formula for the area of a triangle.



So, for a hexagon, for example:

You can see from the diagram that there are six triangles.

The area is:

Height (red line) × length of side (blue line) × 0.5 × 6 (because there are six triangles).

You can also work out the area of any regular polygon using trigonometry, but that's rather more complicated.

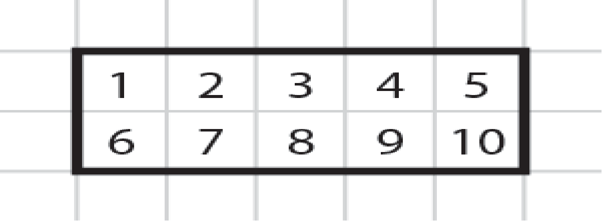
CALCULATING AREA

Area is a measure of how much space there is inside a shape. Calculating the area of a shape or surface can be useful in everyday life – for example you may need to know how much paint to buy to cover a wall or how much grass seed you need to sow a lawn.

This page covers the essentials you need to know in order to understand and calculate the areas of common shapes including squares and rectangles, triangles and circles.

Calculating Area Using the Grid Method

When a shape is drawn on a scaled grid you can find the area by counting the number of grid squares inside the shape.



In this example there are 10 grid squares inside the rectangle.

In order to find an area value using the grid method, we need to know the size that a grid square represents.

This example uses centimetres, but the same method applies for any unit of length or distance. You could, for example be using inches, metres, miles, feet etc.

1 cm square grid

In this example each grid square has a width of 1cm and a height of 1cm. In other words each grid square is one 'square centimetre'.

Count the grid squares inside the large square to find its area..

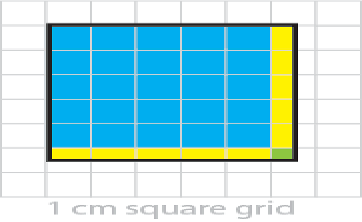
There are 16 small squares so the area of the large square is 16 square centimetres.

In mathematics we abbreviate 'square centimetres' to cm^2. The ^{2} means ‘squared’.

Each grid square is $1cm^2$.

The area of the large square is $16cm^2$.

Counting squares on a grid to find the area works for all shapes – as long as the grid sizes are known. However, this method becomes more challenging when shapes do not fit the grid exactly or when you need to count fractions of grid squares.



In this example the square does not fit exactly onto the grid.

We can still calculate the area by counting grid squares.

- There are 25 full grid squares (shaded in blue).
- 10 half grid squares (shaded in yellow) – 10 half squares is the same as 5 full squares.
- There is also 1 quarter square (shaded in green) – (¼ or 0.25 of a whole square).
- Add the whole squares and fractions together: 25 + 5 + 0.25 = 30.25.

The area of this square is therefore $30.25cm^2$.

You can also write this as $30¼cm^2$.

Although using a grid and counting squares within a shape is a very simple way of learning the concepts of area it is less useful for finding exact areas with more complex shapes, when there may be many fractions of grid squares to add together.

Area can be calculated using simple formulae, depending on the type of shape you are working with.

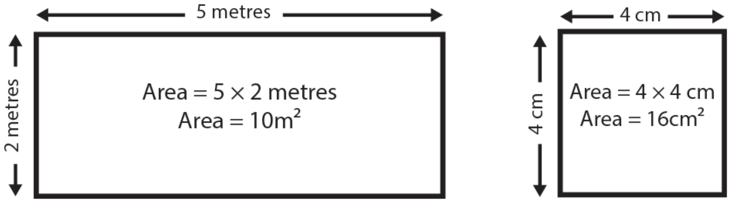
The remainder of this page explains and gives examples of how to calculate the area of a shape without using the grid system.

Areas of Simple Quadrilaterals: Squares and Rectangles and Parallelograms

The simplest (and most commonly used) area calculations are for squares and rectangles. To find the area of a rectangle, multiply its height by its width.

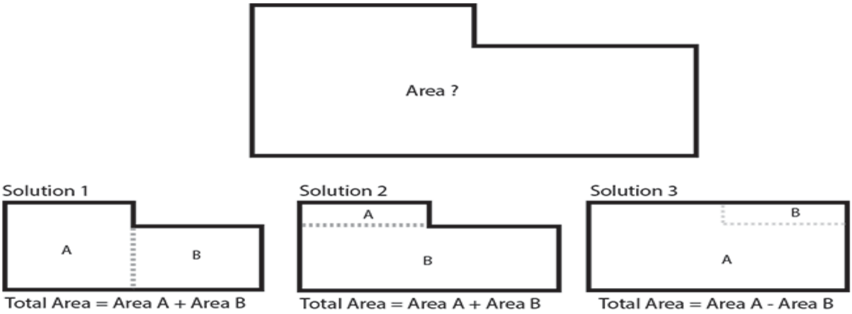
For a square you only need to find the length of one of the sides (as each side is the same length) and then multiply this by itself to find the area. This is the same as saying length^2 or length squared.

It is good practice to check that a shape is actually a square by measuring two sides. For example, the wall of a room may look like a square but when you measure it you find it is actually a rectangle.



Often, in real life, shapes can be more complex. For example, imagine you want to find the area of a floor, so that you can order the right amount of carpet.

A typical floor-plan of a room may not consist of a simple rectangle or square:



In this example, and other examples like it, the trick is to split the shape into several rectangles (or squares). It doesn’t matter how you split the shape - any of the three solutions will result in the same answer.

Solution 1 and 2 require that you make two shapes and add their areas together to find the total area.

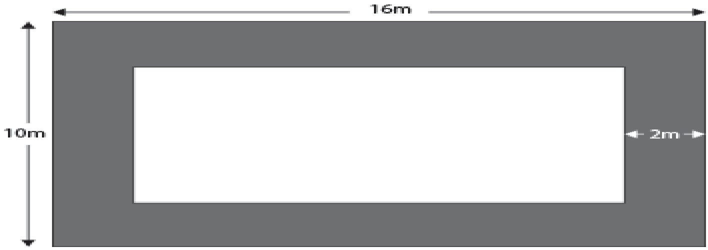
For solution 3 you make a larger shape (A) and subtract the smaller shape (B) from it to find the area.

Another common problem is to the find the area of a border – a shape within another shape.

This example shows a path around a field – the path is 2m wide. Again, there are several ways to work out the area of the path in this example.

You could view the path as four separate rectangles, calculate their dimensions and then their area and finally add the areas together to give a total.

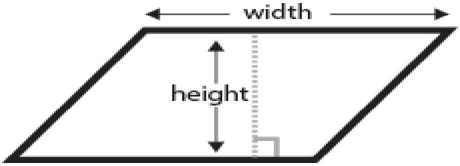
A faster way would be to work out the area of the whole shape and the area of the internal rectangle. Subtract the internal rectangle area from the whole leaving the area of the path.



- The area of the whole shape is 16m × 10m = 160m^2.
- We can work out the dimensions of the middle section because we know the path around the edge is 2m wide.
- The width of the whole shape is 16m and the width of the path across the whole shape is 4m (2m on the left of the shape and 2m on the right). 16m - 4m = 12m
- We can do the same for the height: 10m - 2m - 2m = 6m
- So we have calculated that the middle rectangle is 12m × 6m.
- The area of the middle rectangle is therefore: 12m × 6m = 72m^2.
- Finally we take the area of the middle rectangle away from the area of the whole shape. 160 - 72 = 88m^2.

The area of the path is 88m^2.

A **parallelogram** is a four-sided shape with two pairs of sides with equal length – by definition a rectangle is a type of parallelogram. However, most people tend to think of parallelograms as four-sided shapes with angled lines, as illustrated here.



The area of a parallelogram is calculated in the same way as for a rectangle (height × width) but it is important to understand that height does not mean the length of the vertical (or off vertical) sides but the distance between the sides.

From the diagram you can see that the height is the distance between the top and bottom sides of the shape - not the length of the side.

Think of an imaginary line, at right angles, between the top and bottom sides. This is the height.

Areas of Triangles

It can be useful to think of a triangle as half of a square or parallelogram.

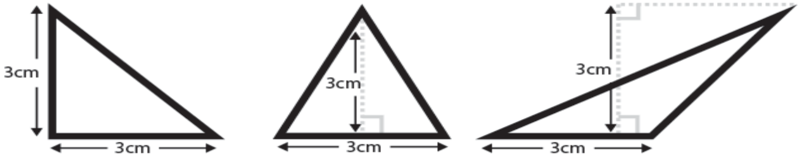
Assuming you know (or can measure) the dimensions of a triangle then you can quickly work out its area.

The area of a triangle is (height × width) ÷ 2.

In other words you can work out the area of a triangle in the same way as the area for a square or parallelogram, then just divide your answer by 2.

The height of a triangle is measured as a right-angled line from the bottom line (base) to the 'apex' (top point) of the triangle.

Here are some examples:



The area of the three triangles in the diagram above is the same.

Each triangle has a width and height of 3cm.

The area is calculated:

(height × width) ÷ 2

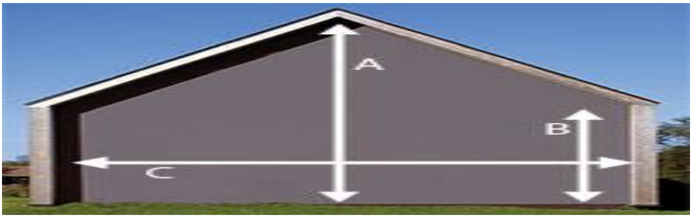
3 × 3 = 9

9 ÷ 2 = 4.5

The area of each triangle is 4.5cm^2.

In real-life situations you may be faced with a problem that requires you to find the area of a triangle, such as:

You want to paint the gable end of a barn. You only want to visit the decorating store once to get the right amount of paint. You know that a litre of paint will cover 10m^2 of wall. How much paint do you need to cover the gable end?



You need three measurements:

A - The total height to the apex of the roof.

B - The height of the vertical walls.

C - The width of the building.

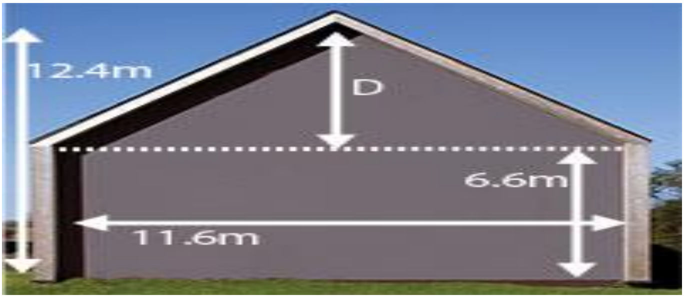
In this example the measurements are:

A - 12.4m

B - 6.6m

C - 11.6m

The next stage requires some additional calculations. Think about the building as two shapes, a rectangle and a triangle. From the measurements you have you can calculate the additional measurement needed to work out the area of the gable end.



Measurement D = 12.4 – 6.6

D = 5.8m

You can now work out the area of the two parts of the wall:

Area of the rectangular part of the wall: 6.6 × 11.6 = 76.56m^2

Area of the triangular part of the wall: (5.8 × 11.6) ÷ 2 = 33.64m^2

Add these two areas together to find the total area:

76.56 + 33.64 = 110.2m^2

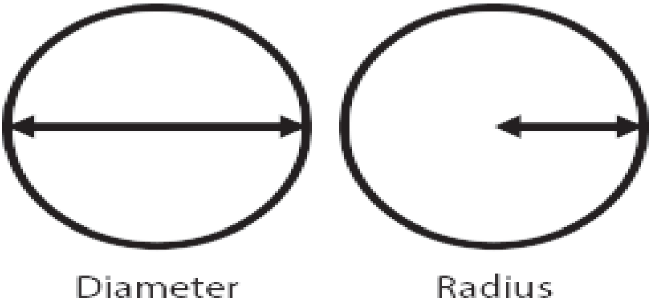
As you know that one litre of paint covers 10m^2 of wall so we can work out how many litres we need to buy:

110.2 ÷ 10 = 11.02 litres.

In reality you may find that paint is only sold in 5 litre or 1 litre cans, the result is just over 11 litres. You may be tempted to round down to 11 litres but, assuming we don't water down the paint, that won't be quite enough. So you will probably round up to the next whole litre and buy two 5 litre cans and two 1 litre cans making a total of 12 litres of paint. This will allow for any wastage and leave most of a litre left over for touching up at a later date. And don't forget, if you need to apply more than one coat of paint, you must multiply the quantity of paint for one coat by the number of coats required!

Areas of Circles

In order to calculate the area of a circle you need to know its **diameter** or **radius**.



The **diameter** of a circle is the length of a straight line from one side of the circle to the other that passes through the central point of the circle. The diameter is twice the length of the radius (diameter = radius × 2)

The **radius** of a circle is the length of a straight line from the central point of the circle to its edge. The radius is half of the diameter. (radius = diameter ÷ 2)

You can measure the diameter or radius at any point around the circle – the important thing is to measure using a straight line that passes through (diameter) or ends at (radius) the centre of the circle.

In practice, when measuring circles it is often easier to measure the diameter, then divide by 2 to find the radius.

You need the radius to work out the area of a circle, the formula is:

circle area = πR^2.

This means:

π = Pi is a constant that equals 3.142.

R - is the radius of the circle.

R^2 (radius squared) means radius × radius.

Therefore a **circle with a radius of 5cm** has an area of:

$3.142 \times 5 \times 5 = 78.55cm^2$.

A **circle with a diameter of 3m** has an area:

First, we work out the radius (3m ÷ 2 = 1.5m)

Then apply the formula:

πR^2

$3.142 \times 1.5 \times 1.5 = 7.0695$.

The area of a circle with a diameter of 3m is 7.0695m^2.

Final Example

This example pulls on much of the content of this page for solving simple area problems.

This is the Ruben M. Benjamin House in Bloomington Illinois, listed on The United States National Register of Historic Places (Record Number: 376599).

This example involves finding the area of the front of the house, the wooden slatted part – excluding the door and windows. The measurements you need are:

A – 9.7m	**B – 7.6m**
C – 8.8m	**D – 4.5m**
E – 2.3m	**F – 2.7m**
G – 1.2m	**H – 1.0m**

Notes:

- All measurements are approximate.
- There is no need to worry about the border around the house – this has not been included in the measurements.
- We assume all rectangular windows are the same size.
- The round window measurement is the diameter of the window.
- The measurement for the door includes the steps.

What is the area of the wooden slatted part of the house?

Workings and answers below:

Answers to above example

First, work out the area of the main shape of the house – that is the rectangle and triangle that make up the shape.

The main rectangle (B × C) 7.6 × 8.8 = 66.88m^2.

The height of the triangle is (A – B) 9.7 – 7.6 = 2.1.

The area of the triangle is therefore (2.1 × C) ÷ 2. 2.1 × 8.8 = 18.48. 18.48 ÷ 2 = 9.24m^2.

The combined full area of the front of the house is the sum of the areas of the rectangle and triangle:

66.88 + 9.24 = 76.12m^2.

Next, work out the areas of the windows and doors, so they can be subtracted from the full area.

The area of the door and steps is (D × E) 4.5 × 2.3 = 10.35m^2.

The area of one rectangular window is (G × F) 1.2 × 2.7 = 3.24m^2.

There are five rectangular windows. Multiply the area of one window by 5.

3.24 × 5 = 16.2m2. (the total area of the rectangular windows).

The round window has a diameter of 1m its radius is therefore 0.5m.

Using πR^2, work out the area of the round window: 3.142 × 0.5 × 0.5 =. 0.7855m^2.

Next add up the areas of the door and windows.

(door area) 10.35 + (rectangle windows area) 16.2 + (round window area) 0.7855 = 27.3355

Finally, subtract the total area for the windows and doors from the full area.

76.12 – 27.3355 = 48.7845

The area of the wooden slatted front of the house, and the answer to the problem is: 48.7845m^2.

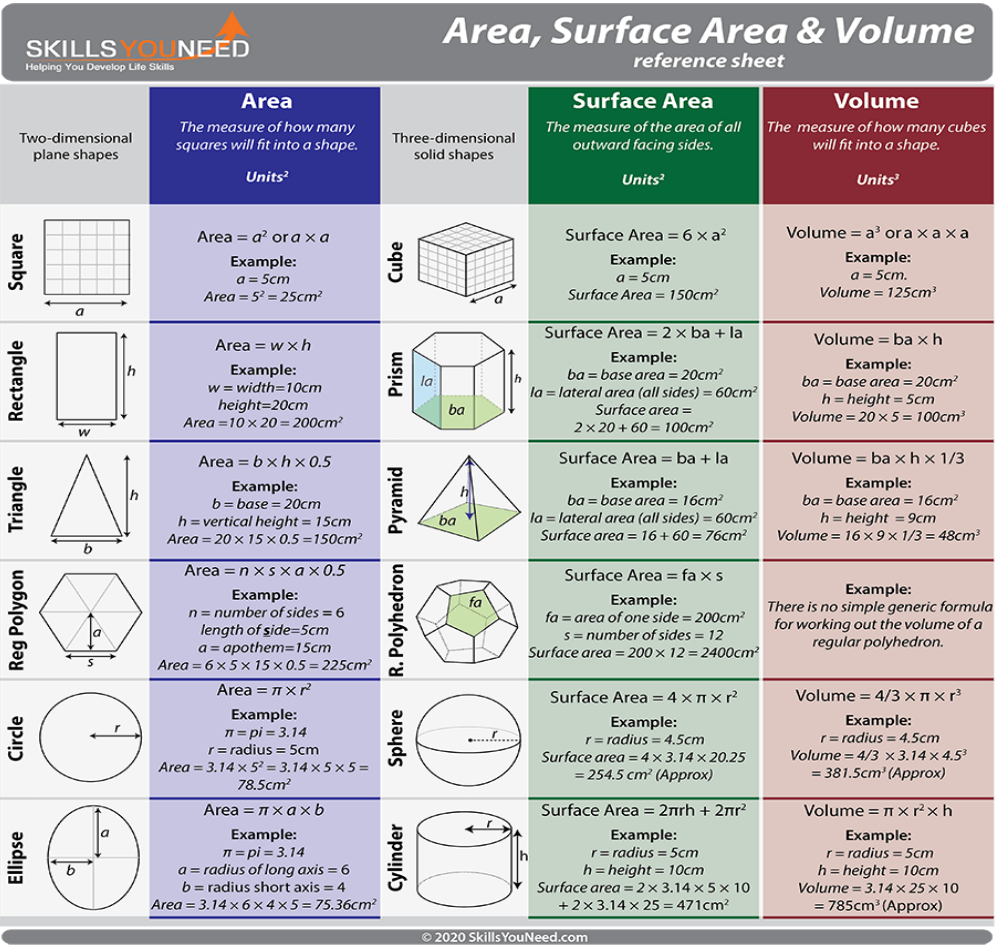
You may want to round the answer up to 48.8m^2 or 49m^2.

Reference.

Read more at: https://www.skillsyouneed.com/num/area.html

Area, Surface Area and Volume

Calculating Area, Three-Dimensional Shapes and Calculating Volume.



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Area, Surface Area & Volume
reference sheet

Two-dimensional plane shapes	**Area** *The measure of how many squares will fit into a shape.* *Units^2*	Three-dimensional solid shapes	**Surface Area** *The measure of the area of all outward facing sides.* *Units^2*	**Volume** *The measure of how many cubes will fit into a shape.* *Units^3*
Square	Area = a^2 or a × a **Example:** *a = 5cm* *Area = 5^2 = 25cm^2*	**Cube**	Surface Area = 6 × a^2 **Example:** *a = 5cm* *Surface Area = 150cm^2*	Volume = a^3 or a × a × a **Example:** *a = 5cm.* *Volume = 125cm^3*
Rectangle	Area = w × h **Example:** *w = width=10cm* *height=20cm* *Area =10 × 20 = 200cm^2*	**Prism**	Surface Area = 2 × ba + la **Example:** *ba = base area = 20cm^2* *la = lateral area (all sides) = 60cm^2* *Surface area =* *2 × 20 + 60 = 100cm^2*	Volume = ba × h **Example:** *ba = base area = 20cm^2* *h = height = 5cm* *Volume = 20 × 5 = 100cm^3*
Triangle	Area = b × h × 0.5 **Example:** *b = base = 20cm* *h = vertical height = 15cm* *Area = 20 × 15 × 0.5 =150cm^2*	**Pyramid**	Surface Area = ba + la **Example:** *ba = base area = 16cm^2* *la = lateral area (all sides) = 60cm^2* *Surface area = 16 + 60 = 76cm^2*	Volume = ba × h × 1/3 **Example:** *ba = base area = 16cm^2* *h = height = 9cm* *Volume = 16 × 9 × 1/3 = 48cm^3*
Reg Polygon	Area = n × s × a × 0.5 **Example:** *n = number of sides = 6* *length of side=5cm* *a = apothem=15cm* *Area = 6 × 5 × 15 × 0.5 = 225cm^2*	**R. Polyhedron**	Surface Area = fa × s **Example:** *fa = area of one side = 200cm^2* *s = number of sides = 12* *Surface area = 200 × 12 = 2400cm^2*	**Example:** *There is no simple generic formula for working out the volume of a regular polyhedron.*
Circle	Area = π × r^2 **Example:** *π = pi = 3.14* *r* = radius = 5cm *Area = 3.14 × 5^2 = 3.14 × 5 × 5 = 78.5cm^2*	**Sphere**	Surface Area = 4 × π × r^2 **Example:** *r = radius = 4.5cm* *Surface area = 4 × 3.14 × 20.25 = 254.5 cm^2 (Approx)*	Volume = 4/3 × π × r^3 **Example:** *r = radius = 4.5cm* *Volume = 4/3 × 3.14 × 4.5^3 = 381.5cm^3 (Approx)*
Ellipse	Area = π × a × b **Example:** *π = pi = 3.14* *a = radius of long axis = 6* *b* = radius short axis = 4 *Area = 3.14 × 6 × 4 × 5 = 75.36cm^2*	**Cylinder**	Surface Area = 2πrh + 2πr^2 **Example:** *r = radius = 5cm* *h = height = 10cm* *Surface area = 2 × 3.14 × 5 × 10* *+ 2 × 3.14 × 25 = 471cm^2*	Volume = π × r^2 × h **Example:** *r = radius = 5cm* *h = height = 10cm* *Volume = 3.14 × 25 × 10* *= 785cm^3 (Approx)*

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Definitions

Apothem: The line connecting the centre of a regular polygon with one of its sides. The line is perpendicular (at a right angle) to the side.

Axis: A line of reference about which an object, point or line is drawn, rotated or measured. In a symmetrical shape, an axis is usually a line of symmetry.

Radius: The distance from the geometric centre of a curved shape to its circumference (edge).

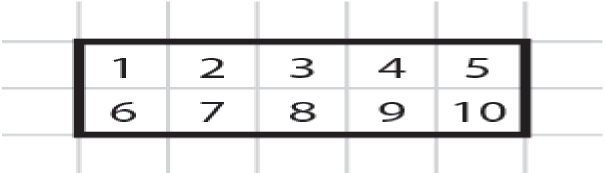
Calculating Area

Area is a measure of how much space there is inside a shape. Calculating the area of a shape or surface can be useful in everyday life – for example you may need to know how much paint to buy to cover a wall or how much grass seed you need to sow a lawn.

This page covers the essentials you need to know in order to understand and calculate the areas of common shapes including squares and rectangles, triangles and circles.

Calculating Area Using the Grid Method

When a shape is drawn on a scaled grid you can find the area by counting the number of grid squares inside the shape.



In this example there are 10 grid squares inside the rectangle.

In order to find an area value using the grid method, we need to know the size that a grid square represents.

This example uses centimetres, but the same method applies for any unit of length or distance. You could, for example be using inches, metres, miles, feet etc.

In this example each grid square has a width of 1cm and a height of 1cm. In other words each grid square is one 'square centimetre'. Count the grid squares inside the large square to find its area..

There are 16 small squares so the area of the large square is 16 square centimetres. In mathematics we abbreviate 'square centimetres' to cm^2. The ^{2} means ‘squared’.

Each grid square is 1cm^2.

The area of the large square is 16cm^2.

Counting squares on a grid to find the area works for all shapes – as long as the grid sizes are known. However, this method becomes more challenging when shapes do not fit the grid exactly or when you need to count fractions of grid squares.

In this example the square does not fit exactly onto the grid.

We can still calculate the area by counting grid squares.

- There are 25 full grid squares (shaded in blue).
- 10 half grid squares (shaded in yellow) – 10 half squares is the same as 5 full squares.
- There is also 1 quarter square (shaded in green) – (¼ or 0.25 of a whole square).

- Add the whole squares and fractions together: 25 + 5 + 0.25 = 30.25.

The area of this square is therefore 30.25cm^2 .

You can also write this as 30¼cm^2.

Although using a grid and counting squares within a shape is a very simple way of learning the concepts of area it is less useful for finding exact areas with more complex shapes, when there may be many fractions of grid squares to add together.

Area can be calculated using simple formulae, depending on the type of shape you are working with.

The remainder of this page explains and gives examples of how to calculate the area of a shape without using the grid system.

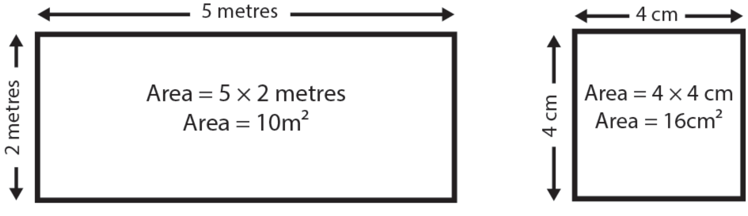
Areas of Simple Quadrilaterals:
Squares and Rectangles and Parallelograms

The simplest (and most commonly used) area calculations are for squares and rectangles.

To find the area of a rectangle, multiply its height by its width.

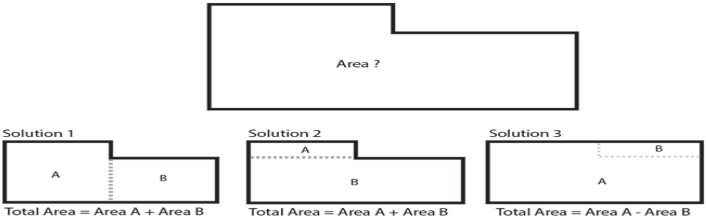
For a square you only need to find the length of one of the sides (as each side is the same length) and then multiply this by itself to find the area. This is the same as saying length^2 or length squared.

It is good practice to check that a shape is actually a square by measuring two sides. For example, the wall of a room may look like a square but when you measure it you find it is actually a rectangle.



Often, in real life, shapes can be more complex. For example, imagine you want to find the area of a floor, so that you can order the right amount of carpet.

A typical floor-plan of a room may not consist of a simple rectangle or square:



In this example, and other examples like it, the trick is to split the shape into several rectangles (or squares). It doesn’t matter how you split the shape - any of the three solutions will result in the same answer.

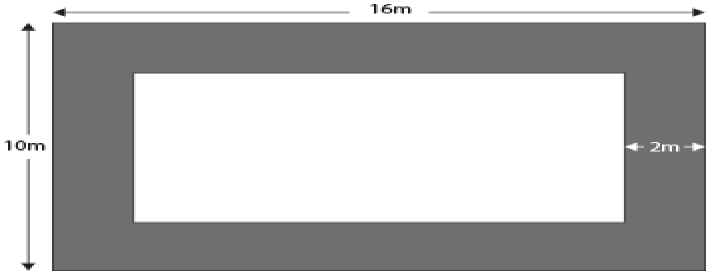
Solution 1 and 2 require that you make two shapes and add their areas together to find the total area. For solution 3 you make a larger shape (A) and subtract the smaller shape (B) from it to find the area.

Another common problem is to the find the area of a border – a shape within another shape.

This example shows a path around a field – the path is 2m wide. Again, there are several ways to work out the area of the path in this example.

You could view the path as four separate rectangles, calculate their dimensions and then their area and finally add the areas together to give a total.

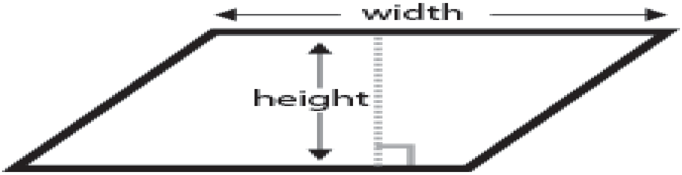
A faster way would be to work out the area of the whole shape and the area of the internal rectangle. Subtract the internal rectangle area from the whole leaving the area of the path.



- The area of the whole shape is 16m × 10m = 160m^2.
- We can work out the dimensions of the middle section because we know the path around the edge is 2m wide.
- The width of the whole shape is 16m and the width of the path across the whole shape is 4m (2m on the left of the shape and 2m on the right). 16m - 4m = 12m
- We can do the same for the height: 10m - 2m - 2m = 6m
- So we have calculated that the middle rectangle is 12m × 6m.
- The area of the middle rectangle is therefore: 12m × 6m = 72m^2.
- Finally we take the area of the middle rectangle away from the area of the whole shape. 160 - 72 = 88m^2.

The area of the path is 88m^2.

A **parallelogram** is a four-sided shape with two pairs of sides with equal length – by definition a rectangle is a type of parallelogram. However, most people tend to think of parallelograms as four-sided shapes with angled lines, as illustrated here.



The area of a parallelogram is calculated in the same way as for a rectangle (height × width) but it is important to understand that height does not mean the length of the vertical (or off vertical) sides but the distance between the sides.

From the diagram you can see that the height is the distance between the top and bottom sides of the shape - not the length of the side.

Think of an imaginary line, at right angles, between the top and bottom sides. This is the height.

Areas of Triangles

It can be useful to think of a triangle as half of a square or parallelogram.

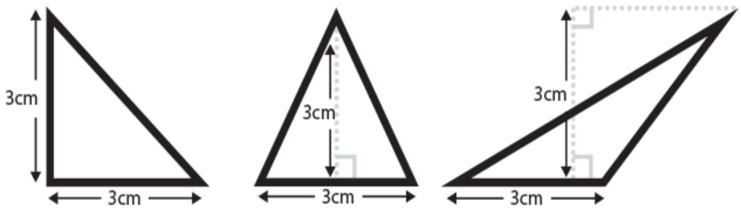
Assuming you know (or can measure) the dimensions of a triangle then you can quickly work out its area.

The area of a triangle is (height × width) ÷ 2.

In other words you can work out the area of a triangle in the same way as the area for a square or parallelogram, then just divide your answer by 2.

The height of a triangle is measured as a right-angled line from the bottom line (base) to the 'apex' (top point) of the triangle.

Here are some examples:



The area of the three triangles in the diagram above is the same.

Each triangle has a width and height of 3cm.

The area is calculated:

(height × width) ÷ 2

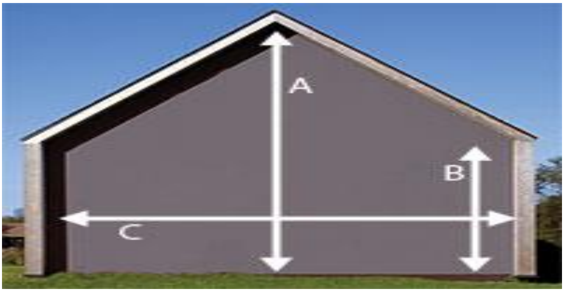
3 × 3 = 9

9 ÷ 2 = 4.5

The area of each triangle is 4.5cm^2.

In real-life situations you may be faced with a problem that requires you to find the area of a triangle, such as:

You want to paint the gable end of a barn. You only want to visit the decorating store once to get the right amount of paint. You know that a litre of paint will cover 10m^2 of wall. How much paint do you need to cover the gable end?



You need three measurements:

A - The total height to the apex of the roof.

B - The height of the vertical walls.

C - The width of the building.

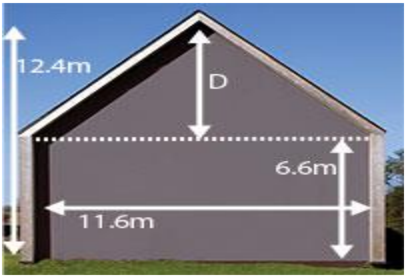
In this example the measurements are:

A - 12.4m

B - 6.6m

C - 11.6m

The next stage requires some additional calculations. Think about the building as two shapes, a rectangle and a triangle. From the measurements you have you can calculate the additional measurement needed to work out the area of the gable end.



Measurement D = 12.4 – 6.6

D = 5.8m

You can now work out the area of the two parts of the wall:

Area of the rectangular part of the wall: 6.6 × 11.6 = 76.56m^2

Area of the triangular part of the wall: (5.8 × 11.6) ÷ 2 = 33.64m^2

Add these two areas together to find the total area:

76.56 + 33.64 = 110.2m^2

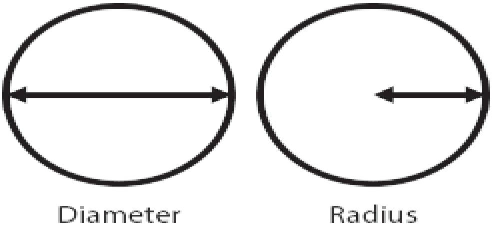
As you know that one litre of paint covers 10m^2 of wall so we can work out how many litres we need to buy:

110.2 ÷ 10 = 11.02 litres.

In reality you may find that paint is only sold in 5 litre or 1 litre cans, the result is just over 11 litres. You may be tempted to round down to 11 litres but, assuming we don't water down the paint, that won't be quite enough. So you will probably round up to the next whole litre and buy two 5 litre cans and two 1 litre cans making a total of 12 litres of paint. This will allow for any wastage and leave most of a litre left over for touching up at a later date. And don't forget, if you need to apply more than one coat of paint, you must multiply the quantity of paint for one coat by the number of coats required!

Areas of Circles

In order to calculate the area of a circle you need to know its **diameter** or **radius**.



The **diameter** of a circle is the length of a straight line from one side of the circle to the other that passes through the central point of the circle. The diameter is twice the length of the radius (diameter = radius × 2)

The **radius** of a circle is the length of a straight line from the central point of the circle to its edge. The radius is half of the diameter. (radius = diameter ÷ 2)

You can measure the diameter or radius at any point around the circle – the important thing is to measure using a straight line that passes through (diameter) or ends at (radius) the centre of the circle.

In practice, when measuring circles it is often easier to measure the diameter, then divide by 2 to find the radius.

You need the radius to work out the area of a circle, the formula is:

circle area = πR^2.

This means:

π = Pi is a constant that equals 3.142.

R = is the radius of the circle.

R^2 (radius squared) means radius × radius.

Therefore a **circle with a radius of 5cm** has an area of:

$3.142 \times 5 \times 5 = 78.55cm^2$.

A **circle with a diameter of 3m** has an area:

First, we work out the radius (3m ÷ 2 = 1.5m)

Then apply the formula:

πR^2

3.142 × 1.5 × 1.5 = 7.0695.

The area of a circle with a diameter of 3m is 7.0695m^2.

Final Example

This example pulls on much of the content of this page for solving simple area problems.

This is the Ruben M. Benjamin House in Bloomington Illinois, listed on The United States National Register of Historic Places (Record Number: 376599).

This example involves finding the area of the front of the house, the wooden slatted part – excluding the door and windows. The measurements you need are:

A – 9.7m	**B – 7.6m**
C – 8.8m	**D – 4.5m**
E – 2.3m	**F – 2.7m**

G – 1.2m	**H – 1.0m**

Notes:

- All measurements are approximate.
- There is no need to worry about the border around the house – this has not been included in the measurements.
- We assume all rectangular windows are the same size.
- The round window measurement is the diameter of the window.
- The measurement for the door includes the steps.

What is the area of the wooden slatted part of the house?

Workings and answers below:

Answers to above example

First, work out the area of the main shape of the house – that is the rectangle and triangle that make up the shape.

The main rectangle (B × C) 7.6 × 8.8 = 66.88m^2.

The height of the triangle is (A – B) 9.7 – 7.6 = 2.1.

The area of the triangle is therefore (2.1 × C) ÷ 2. 2.1 × 8.8 = 18.48. 18.48 ÷ 2 = 9.24m^2.

The combined full area of the front of the house is the sum of the areas of the rectangle and triangle:

66.88 + 9.24 = 76.12m^2.

Next, work out the areas of the windows and doors, so they can be subtracted from the full area.

The area of the door and steps is (D × E) 4.5 × 2.3 = 10.35m^2.

The area of one rectangular window is (G × F) 1.2 × 2.7 = 3.24m^2.

There are five rectangular windows. Multiply the area of one window by 5.

3.24 × 5 = 16.2m2. (the total area of the rectangular windows).

The round window has a diameter of 1m its radius is therefore 0.5m.

Using πR^2, work out the area of the round window: 3.142 × 0.5 × 0.5 =. 0.7855m^2.

Next add up the areas of the door and windows.

(door area) 10.35 + (rectangle windows area) 16.2 + (round window area) 0.7855 = 27.3355

Finally, subtract the total area for the windows and doors from the full area.

76.12 – 27.3355 = 48.7845

The area of the wooden slatted front of the house, and the answer to the problem is: 48.7845m^2.

You may want to round the answer up to 48.8m^2 or 49m^2.

Read more at: https://www.skillsyouneed.com/num/area.html

AREA, SURFACE AREA AND VOLUME REFERENCE SHEET

The graphic on this page, is designed to be a quick reference for calculating the area, surface area and volume of common shapes.

For more information and examples of these calculations see our pages:

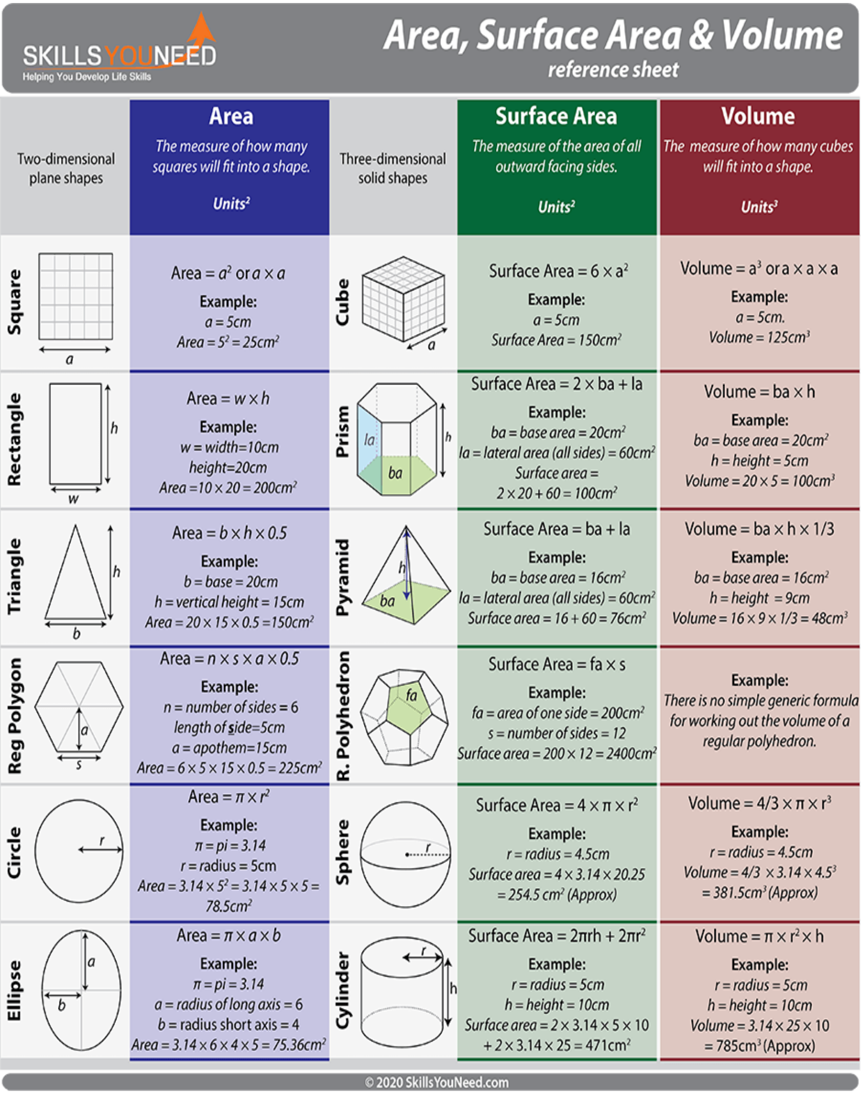
Calculating Area, Three-Dimensional Shapes and Calculating Volume.

Definitions

Apothem: The line connecting the centre of a regular polygon with one of its sides. The line is perpendicular (at a right angle) to the side.

Axis: A line of reference about which an object, point or line is drawn, rotated or measured. In a symmetrical shape, an axis is usually a line of symmetry.

Radius: The distance from the geometric centre of a curved shape to its circumference (edge).



SKILLSYOUNEED
Helping You Develop Life Skills

Area, Surface Area & Volume

reference sheet

Two-dimensional plane shapes	**Area** *The measure of how many squares will fit into a shape.* ***Units^2***	Three-dimensional solid shapes	**Surface Area** *The measure of the area of all outward facing sides.* ***Units^2***	**Volume** *The measure of how many cubes will fit into a shape.* ***Units^3***
Square	$Area = a^2$ or $a \times a$ **Example:** $a = 5cm$ $Area = 5^2 = 25cm^2$	**Cube**	$Surface\ Area = 6 \times a^2$ **Example:** $a = 5cm$ $Surface\ Area = 150cm^2$	$Volume = a^3$ or $a \times a \times a$ **Example:** $a = 5cm.$ $Volume = 125cm^3$
Rectangle	$Area = w \times h$ **Example:** $w = width = 10cm$ $height = 20cm$ $Area = 10 \times 20 = 200cm^2$	**Prism**	$Surface\ Area = 2 \times ba + la$ **Example:** $ba = base\ area = 20cm^2$ $la = lateral\ area\ (all\ sides) = 60cm^2$ $Surface\ area =$ $2 \times 20 + 60 = 100cm^2$	$Volume = ba \times h$ **Example:** $ba = base\ area = 20cm^2$ $h = height = 5cm$ $Volume = 20 \times 5 = 100cm^3$
Triangle	$Area = b \times h \times 0.5$ **Example:** $b = base = 20cm$ $h = vertical\ height = 15cm$ $Area = 20 \times 15 \times 0.5 = 150cm^2$	**Pyramid**	$Surface\ Area = ba + la$ **Example:** $ba = base\ area = 16cm^2$ $la = lateral\ area\ (all\ sides) = 60cm^2$ $Surface\ area = 16 + 60 = 76cm^2$	$Volume = ba \times h \times 1/3$ **Example:** $ba = base\ area = 16cm^2$ $h = height = 9cm$ $Volume = 16 \times 9 \times 1/3 = 48cm^3$
Reg Polygon	$Area = n \times s \times a \times 0.5$ **Example:** $n = number\ of\ sides = 6$ $length\ of\ side = 5cm$ $a = apothem = 15cm$ $Area = 6 \times 5 \times 15 \times 0.5 = 225cm^2$	**R. Polyhedron**	$Surface\ Area = fa \times s$ **Example:** $fa = area\ of\ one\ side = 200cm^2$ $s = number\ of\ sides = 12$ $Surface\ area = 200 \times 12 = 2400cm^2$	**Example:** *There is no simple generic formula for working out the volume of a regular polyhedron.*
Circle	$Area = \pi \times r^2$ **Example:** $\pi = pi = 3.14$ $r = radius = 5cm$ $Area = 3.14 \times 5^2 = 3.14 \times 5 \times 5 =$ $78.5cm^2$	**Sphere**	$Surface\ Area = 4 \times \pi \times r^2$ **Example:** $r = radius = 4.5cm$ $Surface\ area = 4 \times 3.14 \times 20.25$ $= 254.5\ cm^2$ (Approx)	$Volume = 4/3 \times \pi \times r^3$ **Example:** $r = radius = 4.5cm$ $Volume = 4/3 \times 3.14 \times 4.5^3$ $= 381.5cm^3$ (Approx)
Ellipse	$Area = \pi \times a \times b$ **Example:** $\pi = pi = 3.14$ $a = radius\ of\ long\ axis = 6$ $b = radius\ short\ axis = 4$ $Area = 3.14 \times 6 \times 4 \times 5 = 75.36cm^2$	**Cylinder**	$Surface\ Area = 2\pi rh + 2\pi r^2$ **Example:** $r = radius = 5cm$ $h = height = 10cm$ $Surface\ area = 2 \times 3.14 \times 5 \times 10$ $+ 2 \times 3.14 \times 25 = 471cm^2$	$Volume = \pi \times r^2 \times h$ **Example:** $r = radius = 5cm$ $h = height = 10cm$ $Volume = 3.14 \times 25 \times 10$ $= 785cm^3$ (Approx)

COORDİNATES OF A POİNT ON A STRAİGHT LİNE AND ON A STRAİGHT LİNE.

Straight Lines

Any straight line has an equation of the form

$y=mx+c$,

where m, the gradient, is the height through which the line rises in one unit step in the horizontal direction, and c, the intercept, is the y-coordinate of the point of intersection between the line and the y-axis. This is shown in Figure 1.

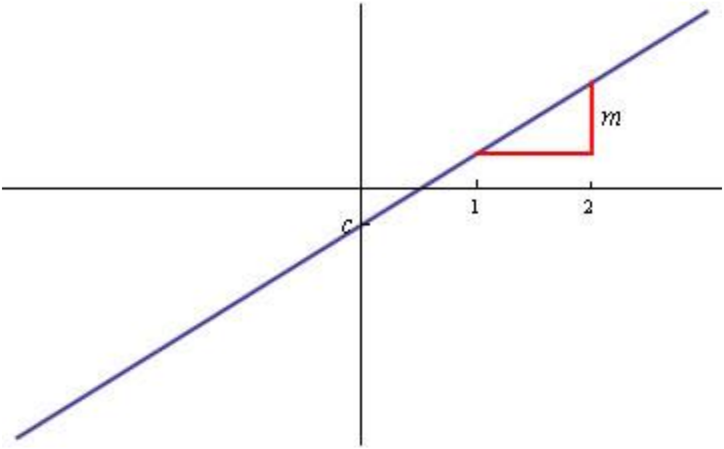


Figure 1: The straight line, $y=mx+c$

If we know the gradient m of a straight line with unknown intercept c, and the coordinates $(x1,y1)$ of a point through which it passes, then we know that

$y1=mx1+c$

and therefore

$c=y1-mx1$.

If we substitute into

$y=mx+c$

we obtain

$y=mx-mx1+y1$

which we can rearrange to give

$y-y1=m(x-x1)$.

So, for example, the straight line through the point $(3,1)$ with gradient 2 is given by

$y-1=2(x-3)$,

which gives

$y=2x-5$.

If we know two points $(x1,y1)$ and $(x2,y2)$ through which passes a line with unknown gradient m and intercept c, then

$y1\ y2 = = mx1+c,\ mx2+c$.

Subtracting the first equation from the second gives

$y2-y1=m(x2-x1)$

and therefore

$m=x2-x1y2-y1$.

The equation of the line is therefore

$y-y1=x2-x1y2-y1(x-x1)$.

So, for example, the straight line through $(-1,-2)$ and $(2,7)$ has equation

$y+2=2+17+2(x+1)$,

which gives

$y=3x+1$.

Finding Points on a Line

To find points on the line $y = mx + b$,

- choose x and solve the equation for y, or
- choose y and solve for x.

Example 1.

Find two points on the line $y = 2x + 1$:

Point 1 - Choose x and solve for y:

Let x =1. Substitute x = 1 into y = 2x + 1 and solve for y.

y = 2(1) + 1 = 2 + 1 = 3

Hence (1,3) is one point on the line y = 2x + 1.

Point 2 - Choose y and solve for x:

Let y = -3. Substitute y = -3 into y = 2x + 1 and solve for x.

-3 = 2x + 1 (Subtract 1 from both sides)

-4 = 2x (Divide both sides by 2)

-2 = X

Hence (-2,-3) is another point on the line y =2x+1.

Exercise 1.

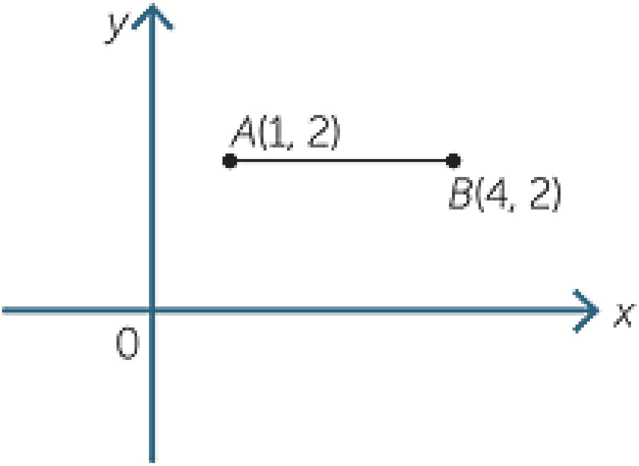
Follow the above example and try some of the following exercises:

EXAMPLE

Find the distance between the following pairs of points.

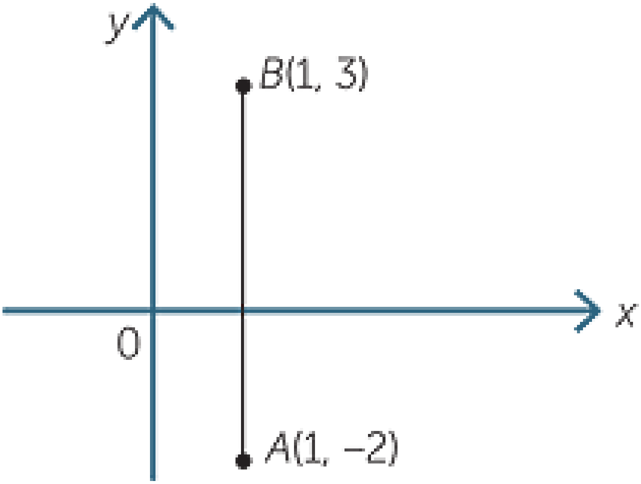
a A(1, 2) and B(4, 2) **b** A(1, −2) and B(1, 3)

SOLUTION



a The distance AB = 4 − 1 = 3

Note: The distance AB is obtained from the difference of the x-coordinates of the two points.



b The distance AB = 3 − (−2) = 5

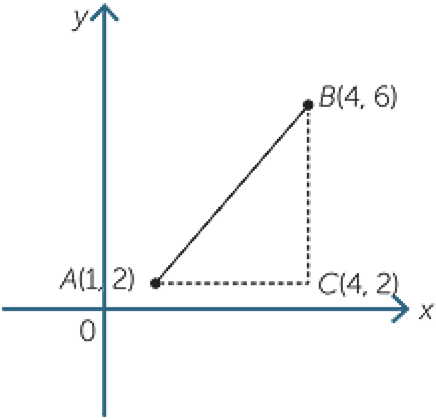
Note: The distance AB is obtained from the difference of the y-coordinates of the two points.

§28. Distance between two points

Distances are always positive, or zero if the points coincide. The distance from A to B is the same as the distance from B to A. We first find the distance between two points that are either vertically or horizontally aligned.

The example above considered the special cases when the line interval AB is either horizontal or vertical. Pythagoras' theorem is used to calculate the distance between two points when the line interval between them is neither vertical nor horizontal.

The distance between the points A(1, 2) and B(4, 6) is calculated below.



AC = 4 − 1 = 3 and BC = 6 − 2 = 4.

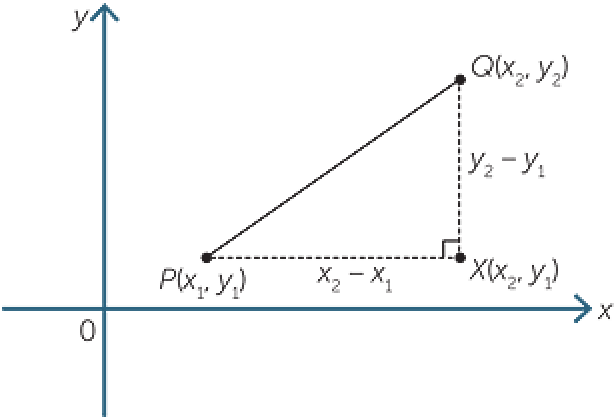
By Pythagoras' theorem,

$$AB^2 = 3^2 + 4^2 = 25$$

And so $AB = 5$

The general case

We can obtain a formula for the length of any interval. Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points.



Form the right-angled triangle PQX, where X is the point (x_2, y_1),

$PX = x_2 - x_1$ or $x_1 - x_2$ and $QX = y_2 - y_1$ or $y_1 - y_2$

depending on the positions of P and Q.

By Pythagoras' theorem:

$$PQ^2 = PX^2 + QX^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Therefore $PQ = QP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Note that $(x_2 - x_1)^2$ is the same as $(x_1 - x_1)^2$ and therefore it doesn't matter whether we go from P to Q or from Q to P − the result is the same.

EXAMPLE

Find the distance between the points A(−4, −3) and B(5, 7).

SOLUTION

In this case, $x_1 = -4$, $x_2 = 5$, $y_1 = -3$ and $y_2 = 7$.

$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$= (5 - (-4))^2 + (7 - (-3))^2$$
$$= 9^2 + 10^2$$
$$= 181$$

Thus, $AB = \sqrt{181}$

Note that we could have chosen $x_1 = 5$, $x_2 = -4$, $y_1 = 7$ and $y_2 = -3$ and still obtained the same result. As long as (x_1, y_1) refers to one point and (x_2, y_2) the other point, it does not matter which one is which.

EXERCISE 1

Show that the distance between the points A(a, b) and B(c,d) is the same as thedistance between

- the points P(a, d) and Q(c, b)
- the points U(b, a) and V(d, c)

Illustrate both of these.

EXERCISE 2

The distance between the points (1, a) and (4, 8) is 5. Find the possible values of a and use a diagram to illustrate.

THE MIDPOINT OF AN INTERVAL

The coordinates of the midpoint of a line interval can be found using averages as we will see. We first deal with the situation where the points are horizontally or vertically aligned.

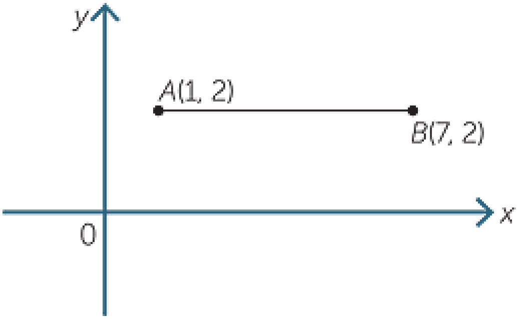
EXAMPLE

Find the coordinates of the midpoint of the line interval AB, given:

a A(1, 2) and B(7, 2)

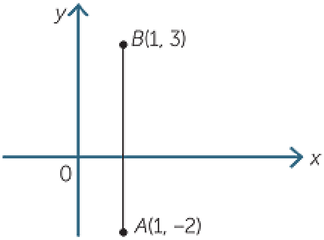
b A(1, −2) and B(1, 3)

SOLUTION



a AB is a horizontal line interval, the midpoint is at (4, 2), since 4 is halfway between 1 and 7.

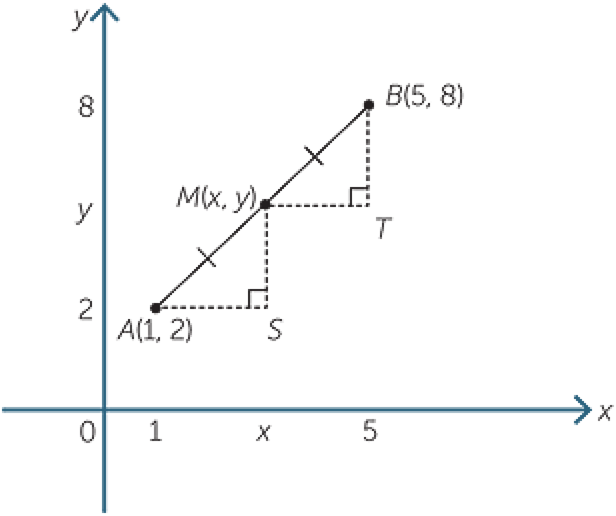
Note: 4 is the average of 1 and 7, that is, 4 = $\frac{1+7}{2}$.



b The midpoint of AB has coordinates 1, $\frac{1}{2}$.

Note that $\frac{1}{2}$ is the average of 3 and −2.

When the interval is not parallel to one of the axes we take the average of the x-coordinate and the y-coordinate. This is proved below.



Let M be the midpoint of the line AB. Triangles AMS and MBT are congruent triangles (AAS), and so AS = MT and MS = BT.

Hence the x-coordinate of M is the average of 1 and 5.

$$x = \frac{5+1}{2} = 3$$

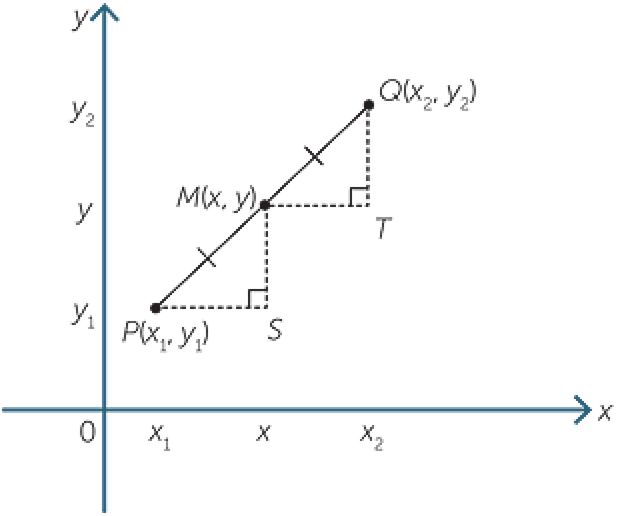
The y coordinate of M is the average of 2 and 8.

$$x = \frac{2+8}{2} = 5$$

Thus the coordinates of the midpoint M are (3, 5).

The general case

We can find a formula for the midpoint of any interval. Suppose that P(x_1, y_1) and Q(x_2, y_2)are two points and let M(x, y) be the midpoint.



Triangles PMS and MQT are congruent triangles (AAS), and so PS = MT and MS = QT.

Hence the x-coordinate of M is the average of x_1 and x_2, and y-coordinate of M is the average of y_1 and y_2. Therefore

$$x = \frac{x_1 + x_2}{2} \text{ and } y = \frac{y_1 + y_2}{2}$$

Midpoint of an interval

The midpoint of an interval with endpoints P(x_1, y_1) and Q(x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Take the average of the x-coordinates and the average of the y-coordinates.

EXAMPLE

Find the coordinates of the midpoint of the line interval joining the points (6, 8) and (−3, 2).

SOLUTION

The midpoint has coordinates, $\left(\frac{6+(-3)}{2}, \frac{8+2}{2}\right) = \left(\frac{3}{2}, 5\right)$

EXAMPLE

If C(3, 6) is the midpoint of line interval AB and A has coordinates (−1, 1), find the coordinates of B.

SOLUTION

Let the coordinates of B be (x_1, y_1).

$\frac{x_1 + (-1)}{2} = 3$ and $\frac{y_1 + 1}{2} = 6$

$x_1 - 1 = 6$ $\quad y_1 + 1 = 12$

so $x_1 = 7$ $\quad$ so $y_1 = 11$.

Thus B has coordinates (7, 11).

EXERCISE 3

A square has vertices O(0, 0), A(a, 0), B(a, a) and C(0, a).

A Find the midpoint of the diagonals OB and CA.

B Find the length of a diagonal of the square and the radius of the circle in which OABC is inscribed. Find the equation of the circle inscribing the square.

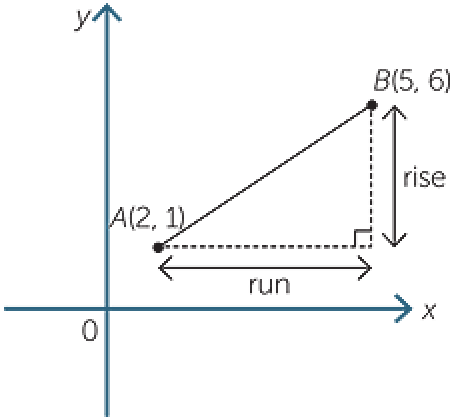
§29. The gradient of a line

Gradient of an interval

The gradient is a measure of the steepness of line. There are several ways to measure steepness. In coordinate geometry the standard way to define the gradient of an interval AB is

$$\frac{\text{rise}}{\text{run}}$$

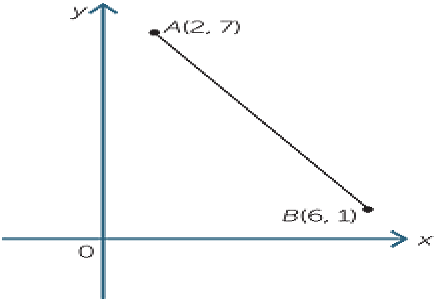
where **rise** is the change in the y-values as you move from A to B and run is the change in the x-values as you move from A to B. We will usually the pronumeral m for gradient.



Given the points A(2, 1) and B(5, 6):

gradient of interval AB = $\frac{\text{rise}}{\text{run}} = \frac{6-1}{5-2} = \frac{5}{3}$

Notice that as you move from A to B along the interval the y-value increases as the x-value increases. The gradient is **positive**.



Given the points A(2, 7) and B(6, 1)

gradient of interval AB = $\frac{\text{rise}}{\text{run}} = \frac{1-7}{6-2} = \frac{-6}{4} = -\frac{3}{2}$

or

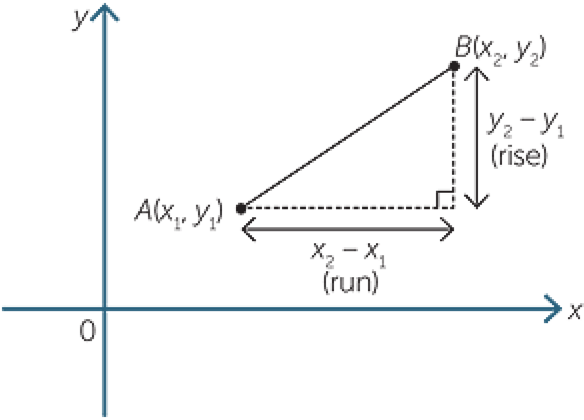
gradient of interval

$$BA = \frac{\text{rise}}{\text{run}} = \frac{7-1}{2-6} = -\frac{3}{2}$$

Notice that in this case as we move from A to B the y value decreases as the x value increases. The gradient is **negative**. Similarly the gradient of

$$BA = -\frac{3}{2}$$

which is the same as the gradient of AB.



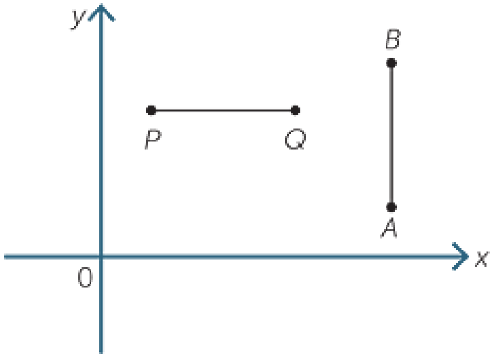
In general:

gradient of line interval

$$AB = \frac{\text{rise}}{\text{run}}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that since $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$ it does not matter which point we take as the first and which point we take as the second.



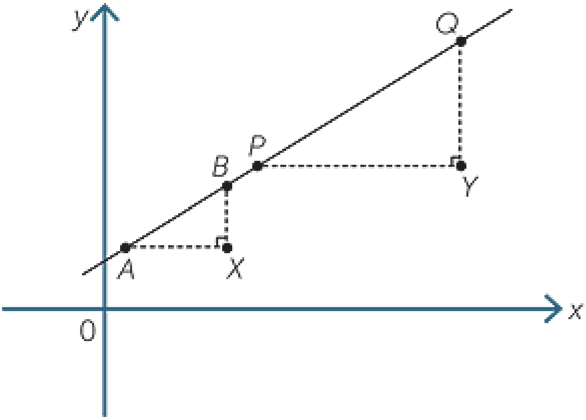
Gradient of *PQ* is zero
Gradient of *AB* is not defined

If the interval is vertical, the run is zero and the gradient of the interval is not defined. This is shown by interval AB. If the interval is horizontal, the rise is zero as shown by interval PQ. The gradient of the interval is zero.

Gradient of a line

The gradient of a line is defined to be the gradient of any interval within the line.

This definition depends on the fact that two intervals on a line have the same gradient.



Suppose AB and PQ are two intervals on the same straight line. Draw right-angled triangles ABX and PQY with sides AX and PY parallel to the x-axis and sides BX and QY parallel to the y-axis.

Triangle ABX is similar to triangle PQY since the corresponding angles are equal. Therefore:

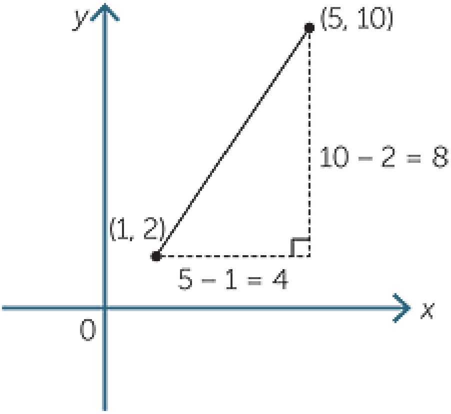
$$\frac{QY}{PY} = \frac{BX}{AX}$$

That is, the intervals have the same gradient.

EXAMPLE

A line passes through the points (1, 2) and (5, 10). Find its gradient.

SOLUTION



$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

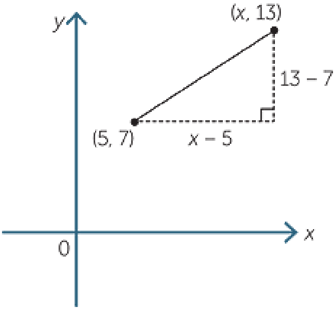
$$= \frac{10 - 2}{5 - 1}$$

$$= 2$$

EXAMPLE

A line passes through the point (5, 7) and has gradient $\frac{2}{3}$. Find the x-coordinate of a point on the line when y = 13.

SOLUTION



Gradient of the line = $\frac{6}{x - 5}$. Thus,

$$\frac{6}{x-5} = \frac{2}{3}$$

$$18 = 2(x - 5)$$

$$9 = x - 5$$

$$x = 14$$

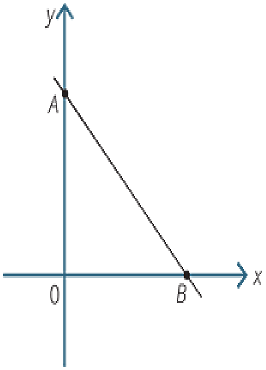
EXERCISE 4

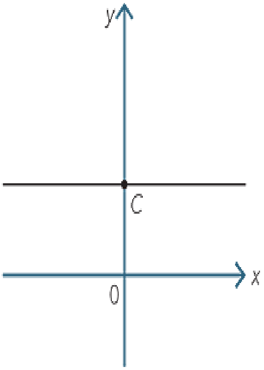
Find the gradient of the line passing through (a, b) and (0, c)

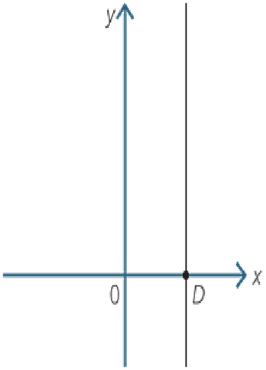
Intercepts

The x-**intercept** of a line is the point at which it crosses the x-axis.

The y-**intercept** of a line is the point at which it crosses the y-axis.





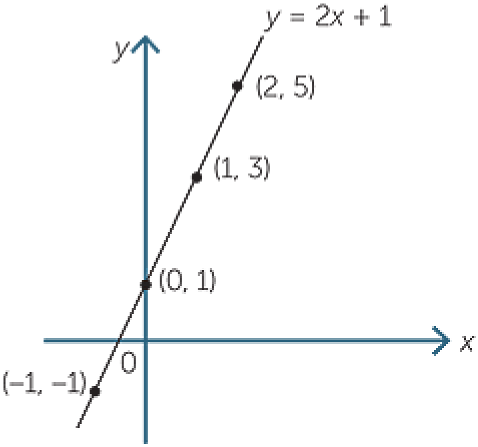


In the diagram to the left the y-intercept is at A and the x-intercept at B.

The second diagram shows a line parallel to the x-axis and it has a y-intercept at C.

The third diagram shows a line parallel to the y-axis and it has an x-intercept at D.

§30. Equation of a straight line

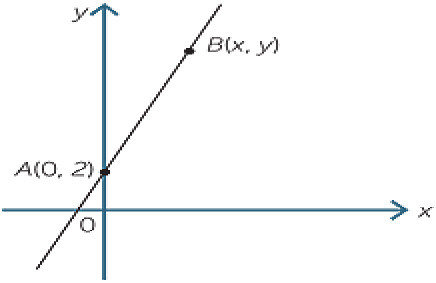


When we plot points which satisfy the equation $y = 2x + 1$ we find that they lie in a straight line.

Can we find the equation of the line given suitable geometric information about the line? The following shows that this can be done given the gradient of the line and the y-intercept.

THE LINE Y = 3X + 2

Consider the line with gradient 3 and y-intercept 2. This passes through the point A(0, 2). Let B(x, y) be any point on this line.



Gradient of interval AB	$= \frac{\text{rise}}{\text{run}}$

	$= \frac{y-2}{x-0}$
	$= \frac{y-2}{x}$

The gradient of the line is 3.

So, $\frac{y-2}{x} = 3$

Rearranging $y - 2 = 3x$

$Y = 3x + 2$

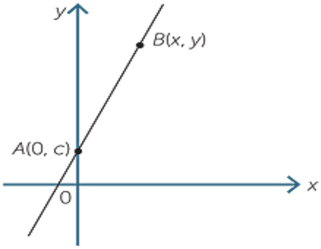
So the coordinates of B(x, y) satisfies y = 3x + 2. This is called the equation of the line.

Conversely suppose that B(x, y) satisfies the equation y = 3x + 2, then $\frac{y-2}{x-0} = 3$ and it passes the point (0, 2) so the point lies on the line with gradient 3 and y-intercept 2.

We summarise this by saying that the equation of the line is y = 3x + 2.

THE EQUATION Y = MX + C

Consider the line with gradient m and y-intercept c. If passes through the point A(0, c). Let B(x, y) be any point on this line.



Gradient of interval AB $= \frac{y-c}{x-0}$

$= \frac{y-c}{x}$

We know the gradient of the line is m.

Therefore $\frac{y-c}{x} = m$

$y - c = mx$

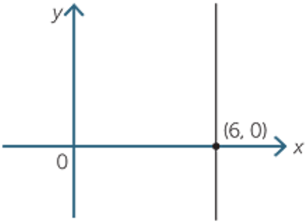
$y = mx + c$

That is, the line in the cartesian plane with gradient m and y-intercept c has equation y = mx + c. Conversely, the points whose coordinates satisfy the equation y = mx + c always lie on the line with gradient m and y-intercept c.

Vertical and horizontal lines

Vertical lines

In a vertical line all points have the same x-coordinate, but the y-coordinate can take any value. The equation of the vertical line through the point (6, 0) is x = 6. The x-axis intercept is 6. All the points on this line have x-coordinate 6.



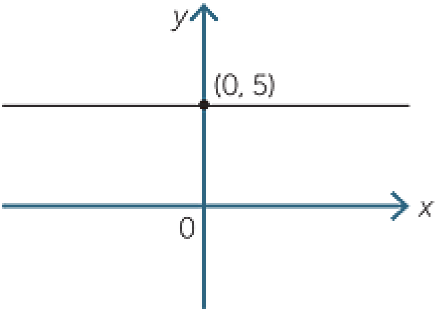
In general, the equation of the vertical line through P(a, b) is x = a.

Because this line does not have a gradient it cannot be written in the form

$y = mx + b.$

Horizontal lines

A horizontal line has gradient 0. In a horizontal line all points have the same y-coordinate, but the x-coordinate can take any value. The equation of the horizontal line through the point (0, 5) is $y = 5$. The equation of the horizontal line through the point (9, 5) is $y = 5$.



In general, the equation of the horizontal line through P(a, b) is $y = b$.

EXAMPLE

Write down the gradient and y-intercept of the line with equation $y = 3x - 4$.

SOLUTION

The gradient of the line is 3 and the y-intercept is −4.

Sometimes an equation needs to be rearranged before the gradient and y-intercept can be determined. Consider the following example.

EXAMPLE

Rewrite the equation $2x + 3y = 6$ in the form $y = mx + c$ and hence find the value of the gradient and y-intercept.

SOLUTION

$2x + 3y = 6$

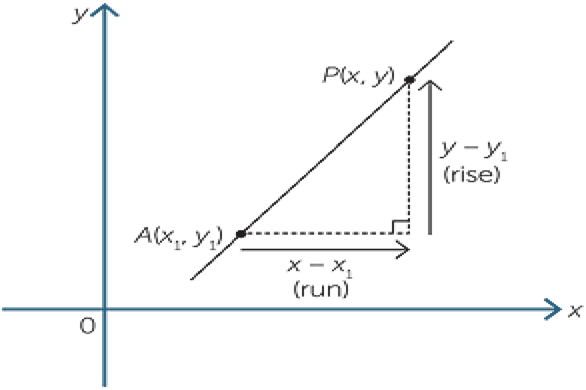
so $3y = 6 - 2x$

that is, $y = 2 - \frac{2x}{3}$

Thus $y = -\frac{2x}{3} + 2$

The gradient of the line is $-\frac{2}{3}$ and the y-intercept is 2.

Equation of a line given its gradient and a point on the line



We want to find the equation of the line with gradient m and which passes through the point $P(x_1, y_1)$.

Let P(x, y) be any point with $x \neq x_1$ on the line passing through the point $A(x_1, y_1)$ and let m be the gradient of this line.

Using gradient, $m = \frac{y - y_1}{x - x_1}$

And $y - y_1 = m(x - x_1)$

This is the equation of the straight line with gradient m passing through the point A(x_1, y_1).

EXAMPLE

Find the equation of the line that passes through the point (−2, 3) with gradient −4.

SOLUTION

The equation for this line is:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -4(x - (-2)) \\ \text{that is,} \quad y - 3 &= -4x - 8 \\ y &= -4x - 5 \end{aligned}$$

Note that it is usual to give the answer in the form y = mx + c

Equation of a straight line given two points

Given two points A(x_1, y_1) and B(x_2, y_2) the equation of the line passing through the two points can be found.

The gradient m of the line passing through A(x_1, y_1) and B(x_2, y_2)

$= \frac{y_2 - y_1}{x_2 - x_1}$, $x_2 \neq x_1$

Substituting into $y - y_1 = m(x - x_1)$ gives

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ , } x_2 \neq x_1$$

GRAPHING STRAIGHT LINES

Using the equation to sketch the line

If you are given an equation of a straight line and asked to draw its graph all you need to do is find two points whose coordinates satisfy the equation and plot the points. There are two commonly used methods to find two points.

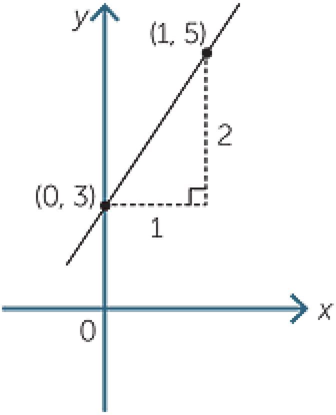
Using the y-intercept and one other point

Using the y-intercept and a second point the equation can be found

EXAMPLE

Draw the graph of y = 2x + 3.

SOLUTION



The y-intercept is 3 and the gradient 2.

Substitute x = 1, so y = 5 giving the point (1, 5) lies on the line.

Plot the two points and draw the line through them.

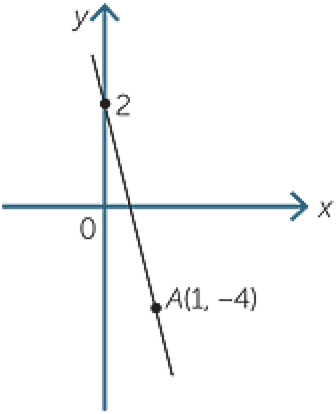
This method does not work if the line is parallel to the y-axis.

EXAMPLE

The gradient of a line is −6 and the y-intercept is 2. Find the equation of the line and sketch it.

SOLUTION

Using the y = mx + c form for the equation of a straight line.



The equation of the line is $y = -6x + 2$.

The point (0, 2) lies on the line.

Substitute $x = 1$ in $y = -6x + 2$

$$y = -6x + 2$$
$$= -6 \times 1 + 2$$
$$= -4$$

The point (1, −4) lies on the line. The graph is shown.

x-intercept y-intercept method

In this method both intercepts are found.

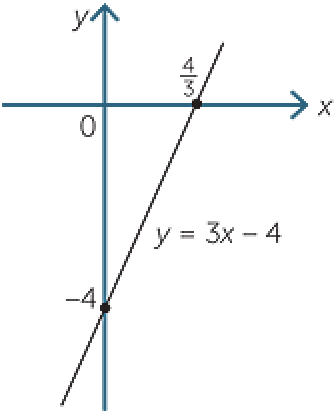
- The x-intercept is found by substituting $y = 0$ and
- The y-intercept is found by substituting $x = 0$.

This method does not work if the line is parallel to an axis or passes through the origin.

EXAMPLE

Using the x-intercept y-intercept method sketch the graph of:

a $y = 3x - 4$ **b** $2x + 3y + 6 = 0$



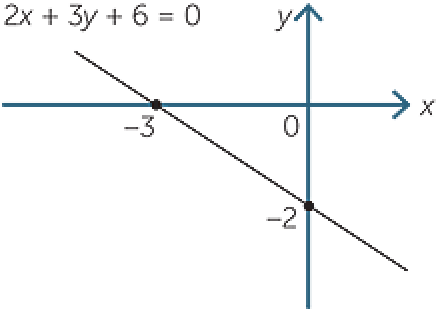
SOLUTION

a When $x = 0$, $y = -4$

When $y = 0$, $3x - 4 = 0$

$3x = 4$

$x = \frac{4}{3}$



b When $x = 0$, $3y + 6 = 0$

$3y = -6$

$y = -2$

When $y = 0$, $2x + 6 = 0$

$$2x = -6$$

$$x = -3$$

The general form for the equation of a straight line

The equations $y = 2x - 3$, $x = 6$ and $2x - 3y = 6$ can be written as $-2x + y + 3$, $x - 6 = 0$ and $2x - 3y - 6 = 0$ respectively.

The **general** form for the equation of a line is $ax + by + c = 0$ where a, b and c are constants and $a \neq 0$ or $b \neq 0$. The equation of every line can be put in general form. The general form is not unique. The equation $x + 2y + 1 = 0$ is the same straight line as $2x + 4y + 2 = 0$.

EXERCISE 5

An equilateral triangle ABC has coordinates O(0, 0), B(a, 0) and C(c, d).

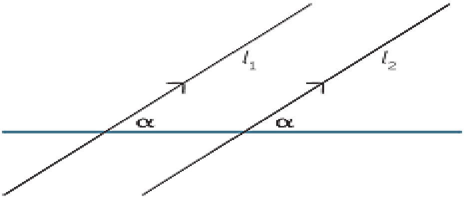
a Find c and d in terms of a by using the fact that OB = BC = CO.

b Find the equation of the lines OB, BC and CO.

PARALLEL AND PERPENDICULAR LINES

Parallel lines

If two lines l_1 and l_2 are parallel then corresponding angles are equal. Conversely, if corresponding angles are equal then the lines are parallel.

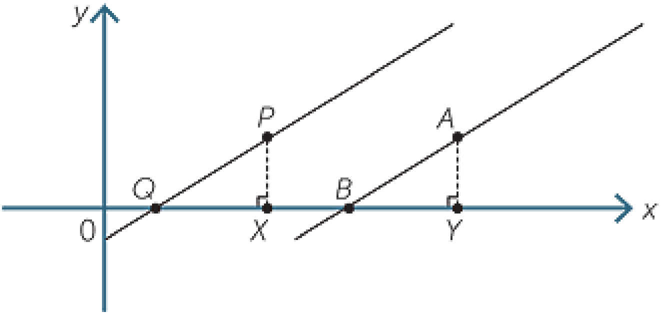


Theorem

Two lines are **parallel** if they have the same gradient and conversely,
two lines with the same gradient are parallel.

Proof

In the diagram, two lines are drawn and the right-angled triangles PQX and ABY are added with QX = BY.



If the lines are parallel then $\angle$PQX = $\angle$ABY (corresponding angles).

The two triangles are congruent by the AAS test.

Therefore PX = PY and $\frac{PX}{QX} = \frac{AY}{BY}$.

That is, the gradients are equal.

Conversely. If the gradients are equal $\frac{PX}{QX} = \frac{AY}{BY}$.

Now QX = BY and therefore PX = AY.

Hence the triangles QPX and ABY are congruent by the SAS test.

Hence the corresponding angles PQX and ABY are equal and the lines are parallel.

EXAMPLE

Show that the line passing through the points A(6, 4) and B(7, 11) is parallel to the line passing through P(0, 0) and Q(2, 14).

SOLUTION

Gradient of AB = $\frac{11-4}{7-6}$ = 7

Gradient of PQ = $\frac{14-0}{2-0}$ = 7

The two lines have the same gradient and so are parallel.

EXAMPLE

Find the equation of the line that is parallel to the line $y = -2x + 6$ and passing through the point A(1, 10).

SOLUTION

The gradient of the line $y = -2x + 6$ is -2.

Therefore the line through the point A(1, 10) parallel to $y = -2x + 6$ has equation:

$$y - y_1 = m(x - x_1)$$

$$y - 10 = -2(x - 1)$$

$$y = -2x + 12$$

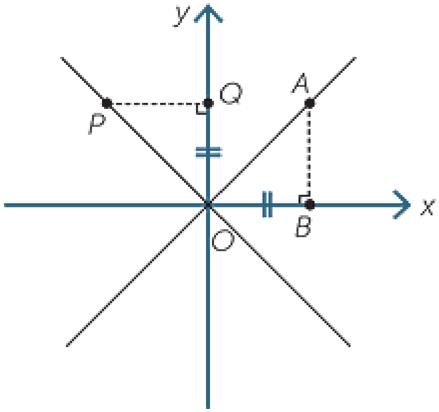
Perpendicular lines

When we draw $y = 3x$ and draw a line perpendicular to it passing through the origin then it is clear that $y = -ax$ where a is a small positive number. We will show that the equation isy $= -\frac{1}{3}x$.

We are now going to show the surprising result that ‘if two lines are **perpendicular** then the product of their gradients is -1’ (or if one is vertical and the other horizontal). The converse is also true. That is, ‘ If the product of the gradients of two lines is -1 then they are perpendicular’.

We first consider the case when both lines pass through the origin.

Draw two lines passing through the origin with one of the lines having positive gradient and the other negative gradient.



Form right-angled triangles OPQ and OAB with OQ = OB.

Gradient of the line OA $= \frac{AB}{OB}$

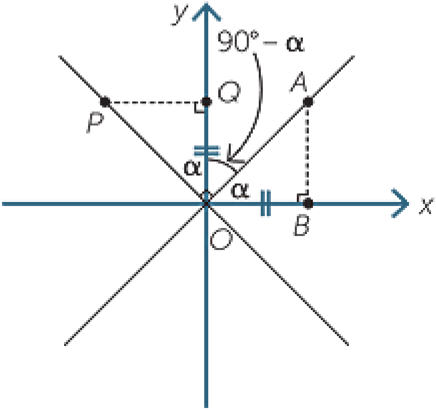
Gradient of the line through OP $= -\frac{OQ}{PQ}$

The product $= -\frac{OQ}{PQ} \times \frac{AB}{OB}$

$= -\frac{OQ}{PQ} \times \frac{AB}{OQ}$ (since OQ = OB)

$= -\frac{AB}{PQ}$

If the lines are perpendicular, ∠POQ = ∠AOB.



Therefore $\triangle$OPQ $\equiv$ $\triangle$OAB (AAS),

so PQ = AB and the product, $-\frac{AB}{PQ}$, of the gradients is −1.

Conversely If the product is −1, then AB = PQ,

so $\triangle$OAB $\equiv$ $\triangle$OPQ (SAS).

Therefore $\angle$POQ = $\angle$AOB and so $\angle$AOP = 90°.

We have now proved the result for lines through the origin.

If we are given two lines anywhere in the plane, we can draw lines through the origin parallel to the given two lines. The gradient of each new line is the same as the gradient of the corresponding original line. So the result holds for lines that do not necessarily pass through the origin.

EXAMPLE

Show that the line through the points A(6, 0) and B(0, 12) is perpendicular to the line through P(8, 10) and Q(4, 8).

SOLUTION

Gradient of AB $= \frac{12 - 0}{0 - 6}$

$= -2$

Gradient of PQ $= \frac{10-8}{8-4}$

$= \frac{2}{4}$

$= \frac{1}{2}$

(Gradient of AB) × (gradient of PQ) $= -2 \times \frac{1}{2}$

$= -1$

Hence the lines are perpendicular.

Parallel and perpendicular lines

- If two non-vertical lines are **parallel** then they have the same gradient. Conversely if two non-vertical lines have the same gradient then they are parallel.
- If two non-vertical lines are **perpendicular** then the product of their gradients is −1. Conversely if the product of the gradients of two lines is −1 then they are perpendicular.

EXAMPLE

Find the equation of the line which passes through the point (1, 3) and is perpendicular to the line whose equation is y = 2x + 1.

SOLUTION

Gradient of the line y = 2x + 1 is 2. Gradient of a line perpendicular to this line is $-\frac{1}{2}$.

The required equation is y – 3	=	$-\frac{1}{2}$ (x − 1)

2(y − 3)	=	−(x − 1)
2y + x	=	7

Thus the equation of the required line is 2y + x = 7.

§31. Proofs with coordinate geometry

Coordinate geometry can be used to prove results in Euclidean Geometry. An important aspect of doing this is placing objects on the Cartesian plane in a way that minimises calculations.

EXAMPLE

Prove that the midpoints of a parallelogram bisect each other using coordinate geometry.

SOLUTION

Let the coordinates of the vertices be O(0, 0), A(a, 0), B(a + c, d) and C(c, d).

There is no loss in generality in placing the vertices of the parallelogram on the Cartesian plane in this way.

The midpoint M of OB = $\left(\frac{a+c}{2}, \frac{d}{2}\right)$

The midpoint N of AC = $\left(\frac{a+c}{2}, \frac{d}{2}\right)$

M = N and so the midpoints coincided which means that the diagonals bisect each other.

EXAMPLE

Prove that the diagonals of a rhombus bisect each other at right angles using coordinate geometry.

SOLUTION

Let the coordinates of the vertices be O(0, 0), A(a, 0), B(a + c, d) and C(c, d).

Because it is a rhombus all the sides are of equal length.

OA = AB = BC = CO

Gradient of OB $=\frac{d}{a+c}$ and,

Gradient of AC $=\frac{d}{c-a}$

The product of the gradients of the diagonals $=\frac{d}{a+c}\times\frac{d}{c-a}$ $=\frac{d^2}{c^2-a^2}$.

$OA^2 = a^2$ and by Pythagoras' theorem, $AB^2 = c^2 + d^2$

so $a^2 = c^2 + d^2$.

Hence $d^2 = -(c^2 - a^2)$

Thus the product of the gradients of the diagonals = −1.

EXERCISE 6

In any triangle ABC prove that $AB^2 + AC^2 = 2(AD^2 + DC^2)$

Where D is the midpoint of BC.

EXERCISE 7

Prove that set of points equidistant from two given points is a straight line.

EXERCISE 8

Prove that the lines joining the midpoints of opposite sides of a quadrilateral and the lines joining the midpoints of its diagonals meet in a point and bisect each other.

Coordinate geometry leads into many other topics in school mathematics. The techniques of coordinate geometry are used in calculus, functions, statistics and many other important areas.

HISTORY

There were three facets of the development of coordinate geometry.

- The invention of a system of coordinates
- The recognition of the correspondence between geometry and algebra
- The graphic representation of relations and functions

The Greek mathematician Menaechmus (380−320 BC) proved theorems by using a method that was very close to using coordinates and it has sometimes been maintained that he had introduced coordinate geometry.

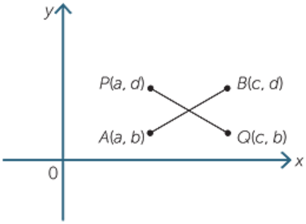
Apollonius of Perga (262−190 BC) dealt with problems in a manner that may be called an coordinate geometry of one dimension; with the question of finding points on a line that were in a ratio to the others. The results and ideas of the ancient Greeks provided a motivation for the development of coordinate geometry.

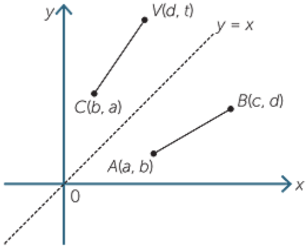
Coordinate geometry has traditionally been attributed to René Descartes (1599−1650) and Pierre de Fermat (1601−1665) who independently provided the beginning of the subject as we know it today.

ANSWERS TO EXERCISES

EXERCISE 1

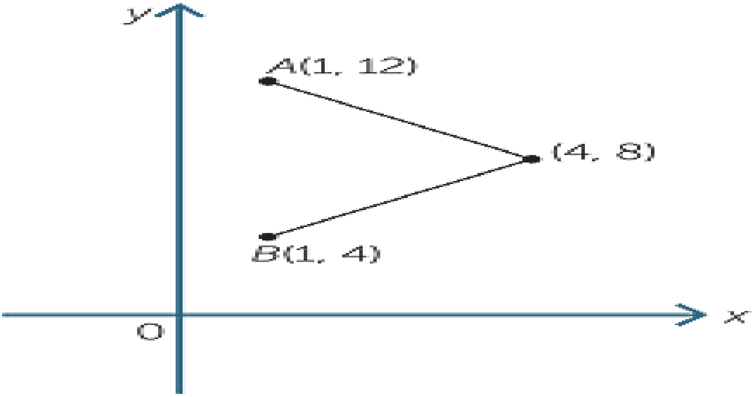
$AB^2 = PQ^2 = UV^2 = (d - b)^2 + (c - a)^2$





EXERCISE 2

a = 4 or a = 12



EXERCISE 3

a $\frac{a}{2}, \frac{a}{2}$

b diagonal = $\sqrt{2}a$, radius = $\frac{\sqrt{2}a}{2}$

c $\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{2}$

EXERCISE 4

$\frac{b - c}{a}$

EXERCISE 5

a $c = \frac{a}{2}$, $d = \frac{\sqrt{3}a}{2}$ **b** OA: $y = 0$, OC: $y = \frac{d}{c}$, OB: $y = \frac{d}{c - a}(x - a)$

EXERCISE 6

Place the triangle so D is at the origin. Then let the coordinates of B and C be (−a, 0) and (a, 0) respectively. Let the coordinates of A be (d, c).

$AB^2 = c^2 + (d + a)^2$ and $AC^2 = c^2 + (d - a)^2$

So $AB^2 + AC^2 = 2c^2 + 2d^2 + 2a^2$

$AD^2 = c^2 + d^2$ and $DC^2 = a^2$

Hence $AB^2 + AC^2 = 2(AD^2 + DC^2)$

EXERCISE 7

Let P(x, y) be a point equidistant from A(a, b) and C(c, d)

PA = PC

$2(c - a)x + 2(d - b)y = d^2 + c^2 - a^2 - b^2$

EXERCISE 8

Let the coordinates of the vertices be O(0, 0), A(a, c), B(m, n) and D(b, 0)

Show that the midpoint of all the required line segments has coordinates

$$\left(\frac{1}{4}(m + b + a), \frac{1}{4}(n + c)\right)$$

§32. Statistical regularities, signs and their classification. Methods of mathematical statistics

The object and method of mathematical statistics.

The statistical description of a collection of objects occupies an intermediate position between the individual description of each object in the collection, on the one hand, and the description of the collection by their common properties, with no individual breakdown into objects, on the other. By comparison with the first method, statistical data are always, to a greater or lesser extent, collective, and have only limited value in cases where the essence is the individual data (for example, a teacher getting to know a class obtains only a very preliminary orientation on the situation from the statistics on the number of excellent, good, adequate, and inadequate appraisals made by his or her predecessor). On the other hand, in comparison with data on a collection which is observed from the outside, and summarized by common properties, statistical data give a deeper penetration into the heart of the matter. For example, data on granulometric analysis of a rock (that is, data on the distribution of rock particles by size) gives valuable additional information when compared to measurements on the unfragmented form of the rock, which allows one, to some extent, to explain the properties of the rock, the conditions of its formation, etc.

The method of research, characterized as the discussion of statistical data on various collections of objects, is called statistical. The statistical method can be applied in very diverse areas of knowledge. However, the features of the statistical method in its applications to various kinds of objects are so specific that it would be meaningless to unify, for example, socio-economic statistics, physical statistics, stellar statistics, etc., in one science.

The common features of the statistical method in various areas of knowledge come down to the calculation of the number of objects in some group or other, the discussion of the distribution of quantitative attributes, the application of the sampling method (in cases where a detailed investigation of an extensive collection is difficult), the use of probability theory to estimate the adequacy of a number of observations for this or that conclusion, etc. This formal mathematical side of statistical research methods is indifferent to the specific nature of the objects being studied and comprises the topic of mathematical statistics.

The connection between mathematical statistics and probability theory.

This connection is different in different cases. Probability theory studies not just any mass phenomenon, but phenomena which are random, to wit, "probabilistically random" . That is, those for which it makes sense to talk of associated probability distributions. Nevertheless, probability theory plays a definite role in the statistical study of mass phenomena of any kind, even those unrelated to the category of probabilistically random phenomena. This comes about through the theories of the sampling method and errors (cf. Errors, theory of; Sample method), which are based on probability theory. In these cases the phenomenon itself is not subject to probabilistic laws, but the means of investigation is.

A more important role is played by probability theory in the statistical investigation of probabilistically random phenomena. Here one finds in full measure the application of such probabilistically based parts of mathematical statistics as statistical hypotheses testing (cf. Statistical hypotheses, verification of), statistical estimation of probability distributions and their parameters, etc. The field of application of these deeper statistical methods is considerably narrower, since it is required that the phenomena themselves are subject to fairly definite probability laws.

For example, the statistical study of turbulent regimes of water flow, or fluctuations in radio reception, is carried out on the basis of the theory of stationary stochastic processes. However, the application of this same theory to the analysis of economic time series may lead to gross errors, since the assumption of a time-invariant probability distribution in the definition of a stationary process is, as a rule, totally unacceptable in this case.

Probability laws gain a statistical expression on the strength of the law of large numbers (probabilities are realized approximately in the form of frequencies, and expectations in the form of averages).

The simplest modes of statistical description.

A collection of nn objects being studied may, relative to some qualitative property AA, be divided into classes A1…ArA1…Ar. The statistical distribution corresponding to this partition is given by the numbers (frequencies) n1…nrn1…nr(where ∑i=1rni=n∑i=1rni=n) of objects in the different classes. Instead of the number nini one often gives the corresponding relative frequency hi=ni/nhi=ni/n(satisfying, obviously, ∑i=1rhi=1∑i=1rhi=1). If the investigation concerns some quantitative attribute, then its distribution in the collection of nn objects may be given by directly listing the observed values of the attribute: x1…xnx1…xn; for example, in increasing order. However, for large nn such a method is cumbersome and, at the same time, does not clearly reveal the essential properties of the distribution. For arbitrarily large nn, in practice it is very unusual to compile complete tables of the observed values xixi, but rather to proceed in all subsequent work from tables which contain only the numbers in the classes obtained by grouping the observations into appropriate intervals.

Usually a grouping into 10–20 intervals, each containing no more than 1515 to 20\pct20\pct of the values xixi, turns out to be sufficient for a fairly complete classification of the essential properties of the distribution and for an appropriate computation, relative to the numbers in the groups, of the basic characteristics of the distribution (see below). Forming a histogram with respect to the grouped data graphically portrays the distribution. A histogram formed on the basis of groups with small intervals obviously has many peaks and does not graphically reflect the essential properties of the distribution. Number of parts. Diameter in mm.

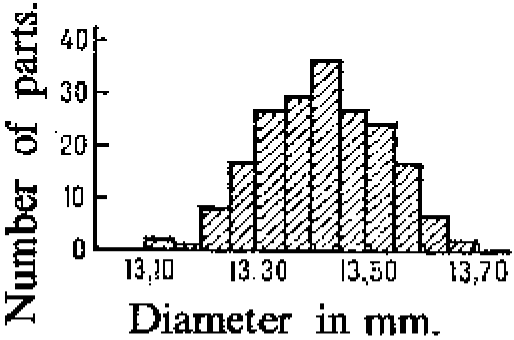


Figure: m062710a

As an example, Fig. ais a histogram for the distribution of 200 diameters of certain parts (in mm), with group intervals of 0.05 mm, and Fig. bis the histogram of the same distribution with intervals of lengths 0.01 mm. Number of parts. Diameter in mm.

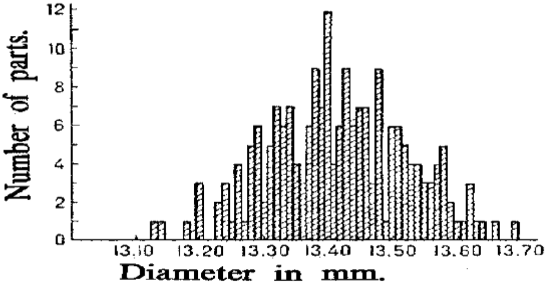


Figure: m062710b

On the other hand, grouping into intervals which are too large may lead to a loss of clarity in the representation of the nature of the distribution, and to gross errors in the calculation of the mean and other characteristics of the distribution (see the corresponding histogram in Fig. c). Number of parts. Diameter in mm.

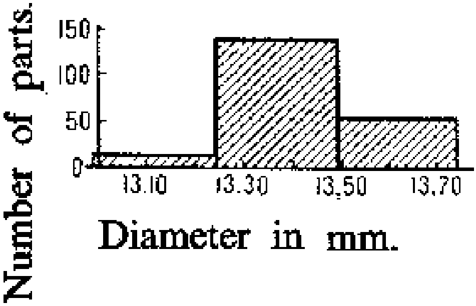


Figure: m062710c

Within the limits of mathematical statistics, questions of grouping into intervals can only be considered from the formal point of view: the completeness of the mathematical description of a distribution, the precision of a calculation of means with respect to grouped data, etc.

The simplest summaries of the characteristics of the distribution of a single quantitative attribute are the mean

x¯¯¯=1n∑i=1nxix¯=1n∑i=1nxi

and the mean-square deviation

D=Sn−−√,D=Sn,

where

S2= ∑i=1n(xi−x¯¯¯)2.S2= ∑i=1n(xi−x¯)2.

In calculating x¯¯¯x¯, S2S2 and DD for grouped data one uses the formulas

x¯¯¯=1n∑k=1rnkak= ∑k=1rhkak,x¯=1n∑k=1rnkak= ∑k=1rhkak,

S2=∑k=1rnk(ak−x¯¯¯)2=∑k=1rnka2k−nx¯¯¯2S2=∑k=1rnk(ak−x¯)2=∑k=1rnkak2−nx¯2 or

D2= ∑k=1rhka2k−x‾‾‾2,D2= ∑k=1rhkak2−x¯2,

where rr is the number of grouped intervals and the akak are their means. If the material is grouped into intervals which are too large, then these calculations are too rough. Sometimes, in such cases it is useful to resort to special refinements of the classification. However, it only makes sense to introduce these refinements when definite probabilistic assumptions are satisfied.

The connection between statistical and probabilistic distributions. Parameter estimators. Testing probabilistic hypotheses.

Above, only certain selected simple modes of statistical description, which form a fairly extensive discipline with a well-developed system of ideas and techniques of calculation, were presented. Modes of statistical description, however, are of interest not just by themselves, but as a means of obtaining, from statistical material, inferences on the laws to which the phenomena studied are subject, and for obtaining inferences on the grounds leading in each individual case to various observed statistical distributions.

For example, the data drawn in Fig. a, Fig. band Fig. cwas collected with the aim of establishing the precision in the manufacturing of parts with design diameter equal to 13.40 mm under normal variations in manufacture. The simplest assumption, which may in this case be based upon some theoretical consideration, is that the diameters of the individual parts can be considered as a random variable XX subject to the normal probability distribution

P{X<x}= 1(2π)1/2σ∫−∞xe−(t−a)2/2σ2dt.(1)(1)P{X<x}= 1(2π)1/2σ ∫−∞xe−(t−a)2/2σ2dt.

If this assumption is true, then the parameters aa and σ2σ2— the mean and the variance of the probability distribution — can be fairly precisely estimated by the corresponding characteristics of the statistical distribution (since the number of observations n=200n=200 is sufficiently large). As an estimator of the theoretical variance it is preferred not to use the statistical variance

D2=S2n,D2=S2n,

but the unbiased estimator

s2=S2n−1.s2=S2n−1.

For the theoretical mean-square deviation σσ there does not exist a single (suitable for any probability distribution) expression of an unbiased estimator. As an estimator (in general, biased) for σσ it is most common to use ss. The accuracy of the estimators x¯¯¯x¯ and ss for aa and σσ is clarified by the corresponding variances, which, in the case of a normal distribution (1), have the form

σ2x¯¯¯= σ2n∼ s2n,σx¯2= σ2n∼ s2n,

σ2s2=2σ4n−1∼ 2s4n,σs22=2σ4n−1∼ 2s4n,

σ2s∼ σ22n∼ s22n,σs2∼ σ22n∼ s22n,

where the sign ∼∼ denotes "approximate equality for large n" . Thus, if one agrees to add to the estimators ±± their mean-square deviation, one has for large nn, under the assumption of a normal distribution (1),

a=x¯¯¯±sn−−√, σ=s±s2–√n.(2)(2)a=x¯±sn, σ=s±s2n.

The sample size n=200n=200 is sufficient for the use in these formulas of laws from the theory of large samples.

For more information on the estimation of the parameters of theoretical probability distributions see Statistical estimation; Confidence estimation.

All rules based on probability theory for the statistical estimation of parameters and hypotheses testing operate only at a definite significance level ω<1ω<1, that is, they may lead to false results with probability α=1−ωα=1−ω. For example, if, under the assumption of a normal distribution and known theoretical variance σ2σ2, an interval estimator of aa based on x¯¯¯x¯ is produced by the rule

x¯¯¯−kσn−−√<a<x¯¯¯+kσn−−√,x¯−kσn<a<x¯+kσn,

then the probability of an error will be equal to αα, which is related to kk through

α= 22π−−√∫k∞e−x2/2dx.α= 22π∫k∞e−x2/2dx.

The question of a rational choice of the significance level under given concrete conditions (for example, in the development of rules for statistical quality control in mass production) is very essential. In this connection the desire to apply only rules with a very high (close to 1) significance level faces the situation that for a restricted number of observations such rules only allow inferences with poor precision (it may not be possible to establish the inequality of probabilities even given a noticeable inequality of the frequencies, etc.).

Further problems in mathematical statistics.

The above-mentioned methods of parameter estimation and hypotheses testing are based on the assumption that the number of observations required to attain a given precision in the conclusions is determined in advance (before carrying out the sampling). However, frequently an a priori determination of the number of observations is inconvenient, since by not fixing the number of trials in advance, but by determining it during the experiment, it is possible to decrease the expected number of trials. This situation was first observed in the example of choosing between one of two hypotheses in a sequence of independent trials.

The corresponding procedure (first proposed in connection with problems of statistical sampling) is as follows: at each step decide, by the results of the observations already carried out, whether to a) conduct the next trial, or b) stop the trials and accept the first hypothesis, or c) stop the trials and accept the second hypothesis. With an appropriate choice of the quantitative characteristics such a procedure can secure (with the same precision in the calculations) a reduction in the average number of observations to almost half that of the fixed size sampling procedure (see Sequential analysis). The development of the methods of sequential analysis led, on the one hand, to the study of controlled stochastic processes (cf. Controlled stochastic process) and, on the other, to the appearance of statistical decision theory.

This theory arises because the results of sequentially carrying out observations serve as a basis for the adoption of certain decisions (intermediate — to continue the trial, and final — when the trials are stopped). In problems on parameter estimation the final decisions are numbers (the values of the estimators), in problems on hypotheses testing they are the accepted hypothesis. The aim of the theory is to give rules for the acceptance of decisions which minimise the mean loss or risk (the risk depends on the probability distributions of the results of the observations, on the final decision, on the expense of conducting the trials, etc.).

Questions on the expedient distribution of effort in carrying out a statistical analysis of phenomena are considered in the theory of design of experiments, which plays a major part in modern mathematical statistics.

Side by side with the development and elaboration of the general ideas of mathematical statistics there have evolved various specialized branches such as dispersion analysis; covariance analysis; multi-dimensional statistical analysis; the statistical analysis of stochastic processes; and factor analysis. New considerations in regression analysis have appeared (see also Stochastic approximation). A major part in problems of mathematical statistics is played by the Bayesian approach to statistical problems.

Historical information.

The first elements of mathematical statistics can already be found in the writings of the originators of probability theory — J. Bernoulli, P. Laplace and S. Poisson. In Russia the methods of mathematical statistics in the application to demography and actuarial work were developed by V.Ya. Bunyakovskii (1846). Of key importance for all subsequent development of mathematical statistics was the work of the classical Russian school of probability theory in the second half of the 19th century and beginning of the 20th century (P.L. Chebyshev, A.A. Markov, A.M. Lyapunov, and S.N. Bernshtein). Many questions of statistical estimation theory were essentially devised on the basis of the theory of errors and the method of least squares (C.F. Gauss and Markov). The work of A. Quételet, F. Galton and K. Pearson has great significance, but in terms of utilizing the achievements of probability theory they lagged behind that of the Russian school. Pearson widely expanded the work on the formation of tables of functions necessary for applying the methods of mathematical statistics. This important work was continued in many scientific centres (in the USSR it was carried out by E.E. Slutskii, N.V. Smirnov and L.N. Bol'shev). In the creation of small sample theory, the general theory of statistical estimation and hypotheses testing (free of assumptions on the presence of a priori distributions), and sequential analysis, the role of the Anglo- American school (Student, the pseudonym of W.S. Gosset, R.A. Fisher, Pearson, and J. Neyman), whose activity began in the 1920's, was very significant. In the USSR noteworthy results in the field of mathematical statistics were obtained by V.I. Romanovskii, A.N. Kolmogorov and Slutskii, to whom belongs important work on the statistics of dependent stationary series, Smirnov, who laid the foundations of the theory of non-parametric methods in statistics, and Yu.V. Linnik, who enriched the analytical apparatus of mathematical statistics with new methods. On the basis of mathematical statistics, statistical methods of research and investigation in queueing theory, physics, hydrology, climatology, stellar astronomy, biology, medicine, etc., were particularly intensively developed.

See also the references to the articles on branches of mathematical statistics.

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§33.Overview of spatial figures (prism, paralelepiped, pyramid, cylinder, cone, sphere, globe). Descriptive illustrations of spatial figures and the discovery of some spatial figures

Planes: parallel, perpendicular, and otherwise

Point, line, and plane are undefined terms. Several assumptions were made about them via the **Point-Line-Plane Postulate** in lesson 1. For planes we must add three more assumptions below.

- **Flat Plane Assumption:** If two points lie in a plane, the line containing them lies in the plane.
- **Unique Plane Assumption:** Through three noncollinear points, there is exactly one plane.
- **Intersecting Planes Assumption:** If two different planes have a point in common, then their intersection is a line.

Planes have no bumps and like lines go on forever. Three (noncollinear) points determine a plane. Three points also determine: a triangle; a line and a point not on the line; and two intersecting lines. Exactly one plane contains these. Thus a three-legged stool is stable, but more legs may cause a chair to wobble. Because lines have no thickness, planes also have no thickness. A line that is not in a plane can intersect the plane in at most one point.

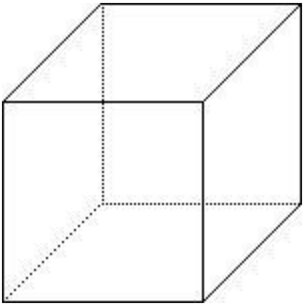
The measure of the smallest of all possible angles defines the angle measure between a line and a plane.

If a line l intersects a plane X at point P, then **line l is perpendicular to plane X** (l ⊥ X) if and only if l is perpendicular to every line in X that contains P.

Line-Plane Perpendicular Theorem: If a line is perpendicular to two different lines at their point of intersection, then it is perpendicular to the plane that contains those lines.

Two planes are **parallel planes** if and only if they have no points in common or they are identical. Again, this inclusive definition is not universally used. The **distance between parallel planes** is the length of a segment perpendicular to the planes with an endpoint in each plane. The **distance between a plane and a point** not on it is measured along the perpendicular segment from the point to the plane.

When two planes intersect, four **dihedral angles** are formed. The measure of these angles can be specified by constructing rays perpendicular to the line of intersection and measuring those angles formed. **Skew lines** are non-coplanar lines which do not intersect.



Just as there is a difference between a polygon and a polygonal region, we differentiate between the surface of a 3-D figure and the space it encloses. With this we start **solid geometry**. A **box**, the surface of a **rectangular solid**, or **parallelepiped** is one of the most important 3-D figures. A box has six **faces** each of which is a rectangular region. **Opposite faces** lie in parallel planes. A **cube** is a box with all faces square regions. The **edges** are line segments where the faces meet each other. The endpoints of the edges are the **vertices**. A box has 12 edges and 8 vertices.

See the figure at the right for a typical 2-D representation and A **tesserack** or **hypercube** is a four dimensional analogue of a cube. See the figure on the left for a 2-D representation of this 4-D object. More information about these can be seen and found. Many people have difficulty believing such can exist which is why such books as Flatland (Abbott, 1884), Sphereland (Burgers, 1983), and Flatterland (Stewart, 2001) were written.

Cylindric solids/surfaces: prisms and cylinders

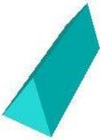
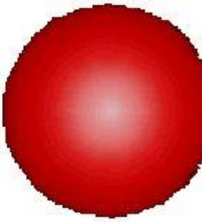
A **cylindric solid** is the set of points between a region and its translation image in space, including the region and its image. A **cylindric surface** is the boundary of a cylindric solid. A **cylinder** is the surface of a cylindric solid with a circular base. A **prism** is the surface of a cylindric solid with a polygonal base.

Cylindric solids have **two bases** which are congruent and in parallel planes. The surface excluding the bases is known as the **lateral surface**. The **height** or **altitude** is the distance between the planes of the bases. If the translation vector is perpendicular to the planes of the bases, the cylindric solid is a **right cylinder** or a **right prism**, otherwise it is **oblique**. Prisms are named by the shape of their base. The non-base faces of a prism are known as **lateral faces** which meet at **lateral edges**. A right prism whose base is a regular polygon is a **regular prism**.

Conic solids/surfaces: pyramids and cones

A **conic solid** is the set of points on any segment between a region (the **base**) and a point (the **vertex**) not in the plane of the base. A **conic surface** is the boundary of a conic solid. A **pyramid** is the surface of a conic solid with a polygonal base. A **cone** is the surface of a conic solid with a circular base.

Conic solids have but **one base**. Pyramids have **lateral edges** which connect vertices of the base polygon with **the** vertex. In a cone, the lateral edge is any segment whose endpoints are the vertex and a point on the base circle. The triangular, non-base, faces of a pyramid are **lateral faces**. Pyramids and cones can also be **right** or **oblique**. A right pyramid with a regular polygonal base is also **regular**. A cone also has an **axis** which is the line (not line segment) though the vertex and the center of the base. The **height** of a conic is the distance between the vertex and the plane containing the base. In a regular pyramid, **slant height** is the altitude of any lateral face (a triangle). Only in a right cone is this the same as the length of a lateral edge. A cone or pyramid may be **truncated**—many drinking glasses are truncated cones. A truncated cone is also known as a **frustum**.

				
Frustum	Pyramid	Prism	Sphere	Torus

Spheres and other things round

A **sphere** is the set of points in space at a certain distance (**radius**) from a point (**center**).

Spheres have **zero bases**. A sphere may be thought of as the three-dimensional analogue of a circle. Twice a radius is a **diameter**. A **Hypersphere** or **4-ball** is a four dimensional analog of a sphere. (Analogue is purposely spelt differently, but also correctly here.) A ball (circle, sphere, hypersphere) in n dimensions is termed an n-ball.

A **plane section** of a three-dimensional figure is the intersection of that figure with a plane.

A sphere and a plane can intersect in very few ways. First, the plane might only touch the sphere at a point. This plane must be tangent at that point thus the line containing the center of the sphere and the point of intersection would be normal (perpendicular) to the plane. Second, the plane intersects in more than one point in which case the intersection is a circle. This intersection is termed a **small circle** unless the plane contains the center of the circle, in which case it is known as a **great circle**. Great circles divide spheres into **hemispheres**, literally half spheres. **Great circle routes** are very important in navigation because they contain the shortest distance between two points on the surface. A **geodesic** is a general term for the shortest distance between two points. **Metric** is the term for this from general relativity. It is how we measure space-time. In differential calculus terms it is: $dx^2+dy^2+dz^2-c^2dt^2$. **Antipode** the exact opposite or contrary; also points of a sphere, such as the earth, which are diametrically opposed (opposite ends of a diameter).

The earth is an **oblate spheroid** (shaped like an M & M, i.e. flattened at the poles and of the three mutually perpendicular radii, the longer is the repeated one). A **prolate spheroid** looks more like a football or cigar (of the three mutually perpendicular radii, the shorter is the repeated one). In an **ellipsoid**, the three mutually perpendicular radii are different lengths. See this site for some diagrams and definitions, including the **torus** or donut.

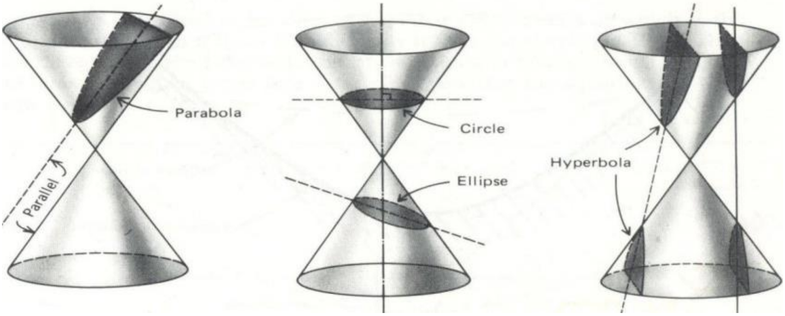
Kepler's Conjecture

By 1606 Kepler started work on what has become known as Kepler's Conjecture after receiving a letter from Harriot who had worked on the problem for at least 15 years. Sir Walter Raleigh actually proposed it to his assistant Harriot. The problem involve the most efficient packing arrangement for cannon balls or, ultimately, atoms in a crystal, although little was known at that time about atomic structure. Ultimately Kepler conjectured that the face centered cubic (FCC) [and equivalently hexagonal close packing (HCP)] both gave the optimal packing density for spheres. Ultimately, this problem became part C of problem 18 on Hilbert's famous list of 23 established in 1900. Until the 1990's, this remained one of the last unsolved problems in discrete geometry. In 1990 Wu-Yi Hsiang sent out preprints and in 1992 submitted for publication a 100 page proof which remains unaccepted in the mathematical community because of its unclarity and lack of logical progression. Although it was published in the International Journal of Mathematics, some have "doubts about the seriousness of the refereeing process" since "the Journal is edited by Hsiang's Berkeley colleagues." It "cannot be considered a proof." (Kepler's

Conjecture, George Szpiro, 2003, page 150.) About 1996 Hales has started a controversial proof which follows a computational approach. On August 9, 1998 Sam Hales announced the availability of papers and preprints proving Kepler's conjecture. In January 1999 a full week workshop dealing solely with Kepler's Conjecture was held at the Institute of Advanced Studies at Princeton where experts scrutized it from every angle. They solicited publication in the Annals of Mathematics. Ultimately it was (will be?) published but with the unprecedented disclaimer that the referees could not verify the proof.

Conic Sections

Nappes is the plural of **Nappe**. A cone is termed double-napped if we are referring to a complete cone which looks more like an hourglass, or two "circular-based pyramids" joined at their vertices. A single "circular-based pyramid" is what most students will think of as a cone.

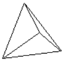
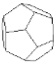


Illustrated above are the intersection of a plane with a cone—a double napped cone. These **loci** (sets of points) are the conic sections. Loci is plural for **locus** (set of points). These conic sections (**circle**, **ellipse**, **parabola**, and **hyperbola**) will be studied more extensively in algebra II. See this web page for further details.

Platonic Solids

A **Platonic Solid** is a convex polyhedra with all faces identical polygons. There are exactly five as proved below.

Platonic Solid

Polyhedron Name	**Face Shape**	**Faces**	**Vertices**	**Edges**	**dual**	**2-D Sketch**	**Ancient**
Tetrahedron	Triangle	4	4	6	self		Fire
Hexahedron	Square	6	8	12	Octahedron		Earth
Octahedron	Triangle	8	6	12	Hexahedron		Air
Dodecahedron	Pentagon	12	20	30	Icosahedron		Cosmos
Icosahedron	Triangle	20	12	30	Dodecahedron		Water

The Platonic Solids are also known as regular polyhedra. A tetrahedron is also known as a **triangular pyramid**. A hexahedron is also known as a **cube**. Duals are especially important in crystallography where the scattered radiation (electrons, neutrons, x-rays) is best studied in reciprocal space. As you look at the table above, please give thought to **Euler's formula** which relates the number of faces, vertices, and edges of any polyhedra: F + V = E + 2.

It can easily be shown that there are only five regular polyhedron (convex and faces all the same regular polygon). Consider how many identical regular polygons can come together at one vertex. We always need more than two if we are going to be able to fold it up and enclose any space. For triangles (with an angle of 60°) six would tesselate the plane. Hence three, four, and five must be considered and the results can be viewed above as the tetrahedron, octahedron, and the icosahedron. For squares (with an angle of 90°) four would tesselate the plane. Hence only three must be considered and the cube is the result. For pentagons (with an angle of 108°) four exceeds 360°. Hence only three must be considered and the dodecahedron is the result. Three hexagons (with an angle of 120°) tesselate the plane. Hence we are have exhausted the possibilities resulting in five regular polyhedra or platonic solids.

This link leads to a page describing the five platonic solids, complete with colored figures. This site has solids you can rotate. This site links to many other good sites. The ancients related the five platonic solids with fire (4), earth (6), air (8), water (20), and the cosmos (12).

Archimedian Solids

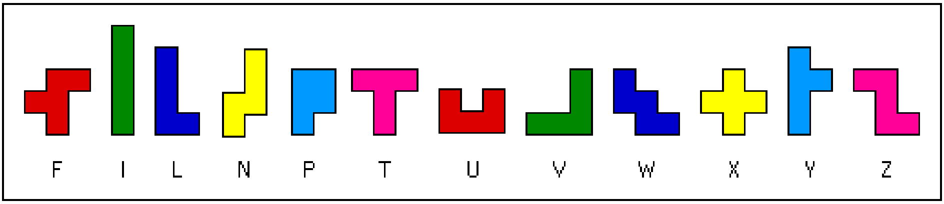
In an **Archimedian Solid**, the solid is convex, all vertices are identical, all faces are regular polygons, but not all identical.

The Archimedian Solids can be classified by a set of numbers which specifies the number of sides in the polygons at each vertex. Thus {3,6,6} would indicate one triangle and two hexagons at each vertex. The other twelve are: {3,8,8}; {4,6,6}; {3,10,10}; {5,6,6}; {3,3,4,4}; {3,3,5,5}; {3,4,4,4}; {4,6,8}; {3,4,4,5}; {4,6,10}; {3,3,3,3,4}; and {3,3,3,3,5}. This (broken) link leads to a page describing the thirteen Archimedian or semi-regular Solids, complete with colored figures.

C_{60} is a highly symmetrical molecule of pure carbon. The shape is the same as that of a soccer ball or the Archimedian Solid the truncated icosahedron: {5,6,6}. C_{60} is often referred to as **Buckyballs.** Technically, this Archimedian Solid is a **Truncated Isocahedron.** This name is derived from Richard Buckminster Fuller, renowned for his **geodesic domes.** The Stepan Center at Notre Dame is a local example. C_{60} is one of a class of compounds known as Fullerenes also name after the American architect above. The C_{60} molecule was discovered in 1985 while a group was trying to understand the absorption spectra of interstellar dust. Their worked earned them the 1996 Nobel prize in chemistry. Initially produced in only tiny amounts, or extracted from soot, it is now readily available and is the center of a lot of varied research. Long tubes of carbon called nanotubes have also been produced. C_{60} represents a new, unexpected crystalline form of solid carbon. The other forms: tetrahedronal carbon bonding in diamond and the sheet type bonding in graphite have much longer histories. See this page for a brief, well documented history of C_{60}.

Pentominoes

The research interests deal with triplication of pentominoes from a subset of 9 of the 12. Can you triplicate all 9 in the set or even all 12? Can you triplicate a given pentomino with all 220 subsets?



Symmetry, Views, and Nets

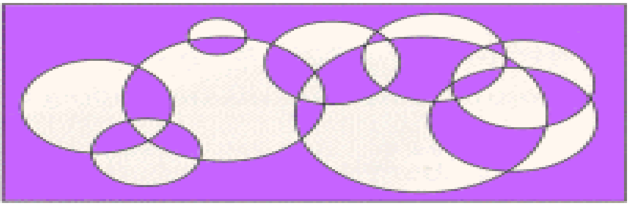
Reflections reverse orientation, thus like 2-D figures, 3-D figures may also be **directly congruent** or **oppositely congruent**. This is especially important in organic chemistry where the orientation of the four bonds around carbon (tetrahedron) is critical to life. See this link for more information (load chime plug-in). I thought in recent years a diet supplement with both isomers was distributed, and many people died, but tryptophan seems to have ultimately had a different (dimer) problem.

Architects often draw scaled **views** or building plans. Some may term these **elevations**. This site has nets for many solids and other use tidbits.

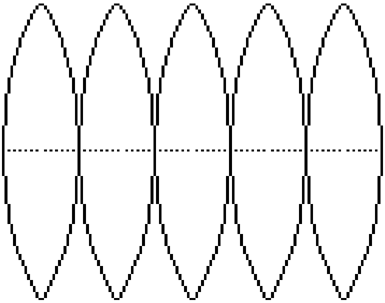
A **net** is a 2-dimensional figure that can be folded on its segments or curved on its boundaries into a 3-dimensional surface.

Map Coloring

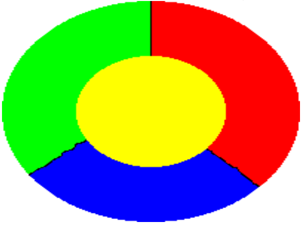
How many colors are necessary and how many colors are sufficient to color any map on a plane? On a sphere (4)? On a Möbius Strip (6)? On a Torus (7)? Maps with boundaries consisting only of straight lines or circles may need even fewer colors! (See figure right.) Except for the plane/sphere, these questions were easy to answer.



Like a net, a map is a 2-D approximation for a 3-D figure. When it is a map of the earth, various distortions are common. If you took a globe, sliced it about every 15° of longitude, the resulting **gores** (see figure left) could be laid flat. This would be fairly accurate, but rather inconvenient—different parts of the same country would be on different gores and the actual shape of these countries difficult to see. Historically, the **Mercator**



projection, created by the Flemish cartographer in 1569, is commonly used. Land areas near the poles are especially distorted—leading to an Africa (11 million miles^2) the size of Greenland (less than 1 million miles^2). This occurs because the Mercator projection is actually the net for a cylinder, not a sphere. The transformation can be generated by extending rays from the center of the earth onto the lateral face of a cylinder. As indicated above, this transformation is not an isometry. However, it does preserve betweenness and collinearity, on the lines of longitude and latitude. Thus the four directions (north, south, east, and west) are on perpendicular lines.



After studying maps, **Francis Guthrie in 1852** conjectured that any map on a sphere or plane could be distinguished/colored with only four colors. It is easy to see that four colors are necessary (see right). However, proof of the sufficiency part was only completed in **1976.** Even then, the proof supplied by **Haken and Appel** was controversial for many years. The controversy arose because they used a computer to help prove only four colors were necessary for each of the 1952 types of possible maps. The problem was embedded in graph theory.